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The lateral migration of spherical particles sedimenting in a stagnant bounded fluid

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Singular perturbation techniques are used to calculate the migration velocity of a spherical particle sedimenting, at low Reynolds numbers, in a stagnant viscous fluid bounded by one or two infinite vertical plane walls. The method is then used to study the migration of a pair of spherical particles sedimenting either in unbounded fluid or in fluid bounded by a plane vertical wall. The migration phenomenon is studied experimentally by recording the trajectory of a spherical particle settling through a viscous fluid bounded by parallel vertical plane walls. Duct- to particle-diameter ratios in the range of 27 to 48 were used with the Reynolds number based on the particle radius being between 0·03 and 0·136.

In all cases the particle is observed to migrate away from the walls until it reaches an equilibrium position at the axis of the duct. The experimentally determined migration velocities agree well with those predicted by the present theory.

1. Introduction

Existing theories for the behaviour of a spherical particle settling, at small Reynolds numbers, in a viscous fluid bounded by vertical plane walls are based on the creeping-motion equations (Faxén 1922; Dean & O'Neill 1963; O'Neill 1964; Goldman, Cox & Brenner 1967*a*, *b*; Cox & Brenner 1967). Although these analyses are able to predict the rotation of and the drag on the particle, they fail to reveal the presence of any force tending to move the particle towards or away from the wall. The absence of such a lift force is a characteristic of the creeping-motion equations and results from the neglect of fluid inertia in the basic equations of motion (Saffman 1956; Bretherton 1962). An attempt to obtain corrections for the inertial effects using the Oseen equations was made by Faxén 1922. However, using the Oseen equations to estimate inertial effects to $O(Re)$ (where Re is the Reynolds number) has been criticized by Proudman & Pearson (1957) and Cox (1965) because these equations do not give the correct asymptotic behaviour of the Navier–Stokes equations to this order in Re.

In this paper, the horizontal migration due to fluid inertia of a spherical particle sedimenting in a stagnant fluid bounded by one or two infinite vertical

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plane walls is considered using the method of matched asymptotic expansions. This technique considers simultaneously two regions of expansions: an inner region surrounding the particle in which viscous effects are dominant and an outer region in which both viscosity and inertia are important (Proudman & Pearson 1957; Rubinow & Keller 1961; Saffman 1965). It follows that, in the present problem, the plane wall can be either in the inner region or in the outer region and different situations arise depending on this choice. The cases in which the fluid is bounded by one and by two plane walls located within the inner region of expansion have been treated respectively by Cox & Hsu (1976) and Vasseur & Cox (1976), the latter situation having also been considered by Ho & Leal (1974). In this investigation the walls are assumed to be located at a large distance from the particle so that they lie within the outer region of expansion. Further, the behaviour of two unequal spherical particles settling, at small Reynolds numbers, in an unbounded fluid is considered and also the behaviour of a single isolated sphere and of two equal-sized spheres sedimenting in a fluid bounded by a single vertical plane wall lying within the outer expansion. Such an investigation is important in determining the effect that the interaction between sedimenting particles has on their average migration velocity. Finally experiments are performed which agree well with the theory.

2. Fundamental equations

Consider a spherical particle, of radius a, sedimenting with a constant velocity V downwards through an incompressible viscous fluid of density ρ and viscosity μ. The fluid is assumed to be at rest far from the sphere and bounded by a rigid plane vertical wall W at a distance d from the particle, the distance d being assumed to be very much larger than the particle radius a (i.e. $a/d \ll 1$). A rectangular Cartesian co-ordinate system (r'_1, r'_2, r'_3) is chosen with its origin at the centre of the sphere and moving with the sphere with the r'_1 axis vertically upwards and the r'_3 axis directed horizontally away from the wall W so that the wall is given by $r'_3 = -d$ (see figure 1). In this co-ordinate system the flow is steady with the fluid at infinity having the velocity $V\mathbf{e}_1$ ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ being unit vectors along the r'_1, r'_2 and r'_3 axes respectively). The sphere is assumed to rotate with an angular velocity $\boldsymbol{\omega}'$.

The velocity $\mathbf{u}'$ and pressure p' (taken to be zero at infinity) in the fluid then satisfy the steady Navier–Stokes and continuity equations, subject to the no-slip boundary condition on the sphere and wall together with appropriate boundary conditions at infinity. Thus

$$\mu\nabla'^2\mathbf{u}' - \nabla' p' = \rho\mathbf{u}'.\nabla'\mathbf{u}', \tag{2.1}$$

$$\nabla'.\mathbf{u}' = 0, \tag{2.2}$$

with the boundary conditions

$$\mathbf{u}' \sim V\mathbf{e}_1 \quad \text{as} \quad r' \to \infty, \quad \mathbf{u}' = V\mathbf{e}_1 \quad \text{on} \quad W, \tag{2.3a}$$

$$\mathbf{u}' = \boldsymbol{\omega}' \times \mathbf{r}' \quad \text{on} \quad r' = a, \tag{2.3b}$$

where $\mathbf{r}' = (r'_1, r'_2, r'_3)$ is the position vector of a general point, and $r' = |\mathbf{r}'|$.

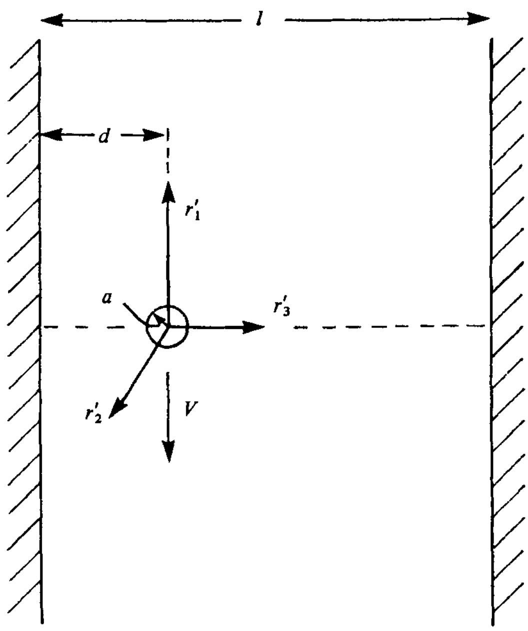

FIGURE 1. Co-ordinate system used. The effect of the second wall is considered in §11.

Introducing the Reynolds number $Re = aV/\nu$ and defining the dimensionless quantities $\mathbf{u}, p, \mathbf{r} = (r_1, r_2, r_3)$ and $\boldsymbol{\omega}$ by

$$\mathbf{u} = \mathbf{u}'/V, \quad p = p'a/\mu V, \quad \mathbf{r} = \mathbf{r}'/a, \quad \boldsymbol{\omega} = \boldsymbol{\omega}'a/V, \tag{2.4}$$

where $\nu = \mu/\rho$ is the fluid kinematic viscosity, (2.1)–(2.3) may be written in non-dimensional form as

$$\nabla^2\mathbf{u} - \nabla p = Re\,\mathbf{u}.\nabla\mathbf{u}, \tag{2.5}$$

$$\nabla.\mathbf{u} = 0, \tag{2.6}$$

with

$$\mathbf{u} \sim \mathbf{e}_1 \quad \text{as} \quad r \to \infty, \quad \mathbf{u} = \mathbf{e}_1 \quad \text{on} \quad W, \tag{2.7a}$$

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{on} \quad r = 1. \tag{2.7b}$$

We shall seek $\mathbf{u}$ and p as expansions in Re valid for

$$Re \ll 1. \tag{2.8}$$

Thus the dimensionless force $\mathbf{F}(= \mathbf{F}'/\mu aV)$ exerted by the fluid on the sphere will be considered as an expansion in terms of this parameter.

3. The inner and outer expansions

The inner or Stokes expansions, valid only in the neighbourhood of the particle where viscous effects are dominant, are

$$\mathbf{u}(Re, \mathbf{r}) = \mathbf{u}_0(\mathbf{r}) + Re\,\mathbf{u}_1(\mathbf{r}) + o(Re), \tag{3.1a}$$

$$p(Re, \mathbf{r}) = p_0(\mathbf{r}) + Re\,p_1(\mathbf{r}) + o(Re) \tag{3.1b}$$

(Proudman & Pearson 1957). Substituting into (2.5), (2.6) and (2.7*b*) and equating powers of Re, one obtains

$$\nabla^2\mathbf{u}_0 - \nabla p_0 = 0, \quad \nabla.\mathbf{u}_0 = 0, \tag{3.2a}$$

$$\mathbf{u}_0 = 0 \quad \text{on} \quad r = 1, \tag{3.2b}$$

and

$$\nabla^2\mathbf{u}_1 - \nabla p_1 = \mathbf{u}_0.\nabla\mathbf{u}_0, \quad \nabla.\mathbf{u}_1 = 0, \tag{3.3a}$$

$$\mathbf{u}_1 = 0 \quad \text{on} \quad r = 1, \tag{3.3b}$$

respectively, where it has been assumed that the sphere does not rotate to these orders in Re. The correctness of this assertion for a freely rotating sphere with no external torque acting on it will be verified later, when it will be shown that the angular velocity of the particle due to the wall is of order Re^2. The conditions imposed on the fields $\mathbf{u}_0$, p_0, $\mathbf{u}_1$ and p_1 are insufficient to determine them uniquely, additional conditions at $r = \infty$ being furnished by the matching of the inner and outer expansions.

For the outer expansions, valid only at large distances from the particle, dimensionless outer variables $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ are defined as

$$\tilde{\mathbf{r}} = Re\,\mathbf{r}. \tag{3.4}$$

These outer expansions are of the form

$$\mathbf{u}(Re, \mathbf{r}) = \mathbf{e}_1 + Re\,\tilde{\mathbf{u}}_1(\tilde{\mathbf{r}}) + o(Re), \tag{3.5a}$$

$$p(Re, \mathbf{r}) = Re^2\tilde{p}_1(\tilde{\mathbf{r}}) + o(Re^2), \tag{3.5b}$$

the first term in the outer expansion of $\mathbf{u}$ being the free-stream velocity $\mathbf{e}_1$.

Rewriting (2.5) and (2.6) in terms of outer variables, substituting the outer expansions into the resulting equations and equating like orders of Re shows that $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ satisfy

$$\tilde{\nabla}^2\tilde{\mathbf{u}}_1 - \tilde{\nabla}\tilde{p}_1 = \partial\tilde{\mathbf{u}}_1/\partial\tilde{r}_1, \quad \tilde{\nabla}.\tilde{\mathbf{u}}_1 = 0, \tag{3.6}$$

which are known as Oseen's equations. From (2.7*a*) and (3.5) the outer boundary conditions satisfied by $\tilde{\mathbf{u}}_1$ are

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r} \to \infty, \quad \tilde{\mathbf{u}}_1 = 0 \quad \text{on} \quad W, \tag{3.7}$$

the inner condition on $\tilde{\mathbf{u}}_1$ being dictated by the requirement that the outer and inner expansions be properly matched.

4. Zeroth-order inner approximation

Matching of outer and inner expansions requires that the inner expansion for $r \to \infty$ when expressed in outer variables is asymptotically identical to the outer expansion for $\tilde{r} \to 0$. This requires that

$$\mathbf{u}_0 \to \mathbf{e}_1 \quad \text{as} \quad r \to \infty. \tag{4.1}$$

The solution of (3.2*a*) subject to the boundary conditions (3.2*b*) and (4.1) is

$$\mathbf{u}_0 = (1 - \tfrac{3}{4}r^{-1} - \tfrac{1}{4}r^{-3})\,\mathbf{e}_1 - \tfrac{3}{4}r_1\,(r^{-3} - r^{-5})\,\mathbf{r}, \quad p_0 = -\tfrac{3}{2}r_1/r^3. \tag{4.2}$$

The force $\mathbf{F}_0$ and torque $\mathbf{G}_0$ (about the sphere's centre) on the sphere due to $\mathbf{u}_0$ are then

$$\mathbf{F}_0 = 6\pi\mathbf{e}_1, \quad \mathbf{G}_0 = 0, \tag{4.3}$$

the former result, when written in dimensional form, giving the well-known drag formula obtained by Stokes (1851). When written in outer variables the velocity $\mathbf{u}_0$ and pressure p_0 take the form

$$\mathbf{u}_0 = \mathbf{e}_1 - Re\frac{3}{4\tilde{r}}\left(\mathbf{e}_1 + \frac{\tilde{r}_1\tilde{\mathbf{r}}}{\tilde{r}^2}\right) + O(Re^3), \quad p_0 = -Re^2\frac{3}{2}\frac{\tilde{r}_1}{\tilde{r}^3}. \tag{4.4}$$

5. First-order inner approximation

The first-order inner flow field $\mathbf{u}_1$, p_1 satisfies (3.3) together with the requirement that it matches onto the outer expansion for $r \to \infty$. One can therefore express this flow field as the sum of two flow fields $\mathbf{u}_{A1}$, p_{A1} and $\mathbf{u}_{B1}$, p_{B1} so that

$$\mathbf{u}_1 = \mathbf{u}_{A1} + \mathbf{u}_{B1} \quad \text{and} \quad p_1 = p_{A1} + p_{B1}, \tag{5.1}$$

where $\mathbf{u}_{A1}$, p_{A1} is any particular solution of (3.3) while $\mathbf{u}_{B1}$, p_{B1} satisfies the homogeneous creeping-flow equations with $\mathbf{u}_{B1} = 0$ on $r = 1$. The velocity field $\mathbf{u}_{A1}$ may be taken to be (Proudman & Pearson 1957)

$$\mathbf{u}_{A1} = \tfrac{3}{32}[(2 - 3r^{-1} + r^{-2} - r^{-3} + r^{-4})(1 - 3r^{-2}r_1^2)\mathbf{r}/r + (4 - 3r^{-1} + r^{-3} - 2r^{-4})r^{-1}r_1(r^{-2}r_1\mathbf{r} - \mathbf{e}_1)], \tag{5.2}$$

which by symmetry cannot give rise to any force or torque on the sphere. For $r \to \infty$

$$\mathbf{u}_{A1} \sim \tfrac{3}{16}[-2r^{-1}r_1\mathbf{e}_1 + (1 - r^{-2}r_1^2)\mathbf{r}/r] + O(r^{-1}), \tag{5.3}$$

which when expressed in outer variables gives

$$Re\,\mathbf{u}_{A1} \sim \tfrac{3}{16}Re[-2\tilde{r}^{-1}\tilde{r}_1\mathbf{e}_1 + (1 - \tilde{r}^{-2}\tilde{r}_1^2)\tilde{\mathbf{r}}/\tilde{r}] + O(Re^2). \tag{5.4}$$

In order to match onto the outer expansion, $\mathbf{u}_{B1}$ must be of order r^0 as $r \to \infty$, since terms like $r^1, r^2, \ldots$ would match onto terms of order Re^0, Re^{-1}, ... in the outer expansion which just do not exist [the term in Re^0 having already been matched onto $\mathbf{u}_0$]. Thus as $r \to \infty$, it may be shown (Brenner & Cox 1963) that $\mathbf{u}_{B1}$ is of the form

$$\mathbf{u}_{B1} = \boldsymbol{\gamma} + O(r^{-1}), \tag{5.5}$$

where $\boldsymbol{\gamma}$ is a constant vector. Since $\mathbf{u}_{B1}$ satisfies the creeping-flow equations with the no-slip boundary condition on the sphere, it gives rise to a force $6\pi\boldsymbol{\gamma}$ and zero torque about the sphere's centre. Thus to order Re the force $\mathbf{F}$ and torque $\mathbf{G}$ on the sphere are

$$\mathbf{F} = 6\pi(\mathbf{e}_1 + Re\,\boldsymbol{\gamma} + o(Re)), \tag{5.6a}$$

$$\mathbf{G} = o(Re). \tag{5.6b}$$

It is therefore seen that any angular velocity the sphere might have owing to the wall in the outer region must be $o(Re)$ as we had assumed. Furthermore as $r \to \infty$ the asymptotic form of $\mathbf{u}$ derived from (4.4), (5.4) and (5.5) when expressed in the outer variables is

$$\mathbf{u} \sim \mathbf{e}_1 + Re[-\tfrac{3}{4}\tilde{r}^{-1}(\mathbf{e}_1 + \tilde{r}_1 \tilde{\mathbf{r}}/\tilde{r}^2) + \tfrac{3}{16}\{-2\tilde{r}^{-1}\tilde{r}_1 \mathbf{e}_1 + (1-\tilde{r}_1^2/\tilde{r}^2)\tilde{\mathbf{r}}/\tilde{r}\} + \boldsymbol{\gamma}] + O(Re^2). \quad (5.7)$$

The value of the arbitrary constant vector $\boldsymbol{\gamma}$ in this expression is determined by matching onto the outer expansion.

6. First-order outer approximation

The first-order outer flow field $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ satisfies (3.6) subject to the boundary condition (3.7). In addition, it is seen from (5.7) that the required matching condition on $\tilde{\mathbf{u}}_1$ as $\tilde{r} \to 0$ is

$$\tilde{\mathbf{u}}_1 \sim -\tfrac{3}{4}\tilde{r}^{-1}(\mathbf{e}_1 + \tilde{r}_1 \tilde{\mathbf{r}}/\tilde{r}^2) + O(\tilde{r}^0), \quad (6.1)$$

which represents the flow field produced by a point force of $-6\pi\mathbf{e}_1$ at $\tilde{r} = 0$. Hence the first-order outer equations and boundary conditions for $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ may be written as

$$\tilde{\nabla}^2\tilde{\mathbf{u}}_1 - \tilde{\nabla}\tilde{p}_1 - \partial\tilde{\mathbf{u}}_1/\partial\tilde{r}_1 = 6\pi\mathbf{e}_1\delta(\tilde{\mathbf{r}}), \quad \tilde{\nabla}\cdot\tilde{\mathbf{u}}_1 = 0, \quad (6.2a)$$

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r} \to \infty, \quad \tilde{\mathbf{u}}_1 = 0 \quad \text{on} \quad W, \quad (6.2b)$$

where $\delta(\tilde{\mathbf{r}})$ is the Dirac delta function.

To solve these equations we introduce $\boldsymbol{\Gamma}$ and Π, the two-dimensional Fourier transforms of the velocity $\tilde{\mathbf{u}}_1$ and pressure $\tilde{p}_1$, defined by

$$\boldsymbol{\Gamma}(k_1, k_2, \tilde{r}_3) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\mathbf{u}}_1(\tilde{\mathbf{r}})\exp[-i(k_1\tilde{r}_1 + k_2\tilde{r}_2)]\,d\tilde{r}_1\,d\tilde{r}_2, \quad (6.3)$$

together with a similar equation for $\Pi(k_1, k_2, \tilde{r}_3)$. $\tilde{\mathbf{u}}_1$ and $\tilde{p}_1$ are then given by inverse Fourier transforms:

$$\tilde{\mathbf{u}}_1(\tilde{\mathbf{r}}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{\Gamma}(k_1, k_2, \tilde{r}_3)\exp[i(k_1\tilde{r}_1 + k_2\tilde{r}_2)]\,dk_1\,dk_2, \quad (6.4)$$

and a similar equation for $\tilde{p}_1(\tilde{\mathbf{r}})$. Taking the Fourier transform of (6.2a) shows that $\boldsymbol{\Gamma}$ and Π satisfy

$$\left\{-k_1^2 - k_2^2 + \frac{\partial^2}{\partial\tilde{r}_3^2}\right\}\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} - \begin{pmatrix}ik_1\\ ik_2\\ \partial/\partial\tilde{r}_3\end{pmatrix}\Pi - ik_1\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} = \frac{1}{4\pi^2}\begin{pmatrix}6\pi\\ 0\\ 0\end{pmatrix}\delta(\tilde{r}_3), \quad (6.5)$$

$$i(k_1\Gamma_1 + k_2\Gamma_2) + \partial\Gamma_3/\partial\tilde{r}_3 = 0. \quad (6.6)$$

Multiplying the first component of (6.5) by ik_1, the second component by ik_2 and adding the resulting equations one obtains

$$(k_1 - iq^2)(k_1\Gamma_1 + k_2\Gamma_2) + i(k_1\Gamma_1'' + k_2\Gamma_2'') + q^2\Pi = (3/2\pi)\,ik_1\delta(\tilde{r}_3), \quad (6.7)$$

where $q^2 = k_1^2 + k_2^2$ and the prime refers to differentiation with respect to $\tilde{r}_3$. Making use of (6.6), one obtains from (6.7) the value of Π as

$$q^2\Pi = \Gamma_3''' - t^2\Gamma_3' + (3/2\pi)\,ik_1\delta(\tilde{r}_3), \quad (6.8)$$

where $t^2 = k_1^2 + k_2^2 + ik_1$. Differentiating (6.8) with respect to $\tilde{r}_3$ and substituting the resulting expression into the third component of (6.5) shows that Γ_3 satisfies

$$\Gamma_3^{\mathrm{iv}} - (2q^2 + ik_1)\,\Gamma_3'' + q^2t^2\Gamma_3 = -(3/2\pi)\,ik_1\,\delta'(\tilde{r}_3). \tag{6.9}$$

This can be integrated and the solution obtained as

$$\Gamma_3 + A\,e^{-q\tilde{r}_3} + B\,e^{-t\tilde{r}_3} + C\,e^{q\tilde{r}_3} + D\,e^{t\tilde{r}_3} + (3/4\pi)\,\mathrm{sgn}\,(\tilde{r}_3)\,(e^{-q|\tilde{r}_3|} - e^{-t|\tilde{r}_3|}), \tag{6.10}$$

where A, B, C and D are constants of integration that have to be evaluated from the boundary conditions. Substituting the value of Γ_3 given by (6.10) into (6.9) one obtains

$$\Pi = ik_1 q^{-1}(A\,e^{-q\tilde{r}_3} - C\,e^{q\tilde{r}_3} + (3/4\pi)\,e^{-q|\tilde{r}_3|}). \tag{6.11}$$

The first and second components of (6.5) can be rewritten as

$$\Gamma_1'' - t^2\Gamma_1 = (3/2\pi)\,\delta(\tilde{r}_3) + ik_1\,\Pi, \tag{6.12}$$

$$\Gamma_2'' - t^2\Gamma_2 = ik_2\,\Pi. \tag{6.13}$$

These ordinary differential equations for Γ_1 and Γ_2 can be integrated using (6.11) to obtain

$$\Gamma_1 = -Aik_1 q^{-1}e^{-q\tilde{r}_3} + G\,e^{-t\tilde{r}_3} + Cik_1 q^{-1}e^{q\tilde{r}_3} + H\,e^{t\tilde{r}_3} - (3/4\pi)\,\{ik_1 q^{-1}e^{-q|\tilde{r}_3|} - (ik_1 - 1)\,t^{-1}e^{-t|\tilde{r}_3|}\}, \tag{6.14}$$

and

$$\Gamma_2 = -Aik_2 q^{-1}e^{-q\tilde{r}_3} + E\,e^{-t\tilde{r}_3} + Cik_2 q^{-1}e^{q\tilde{r}_3} + F\,e^{t\tilde{r}_3} - (3/4\pi)\,\{ik_2 q^{-1}e^{-q|\tilde{r}_3|} - ik_2 t^{-1}e^{-t|\tilde{r}_3|}\}, \tag{6.15}$$

where E, F, G and H are constants of integration.

Substituting (6.14), (6.15) and (6.10) into the continuity equation (6.6) shows that

$$ik_1 G + ik_2 E - Bt = 0, \quad ik_1 H + ik_2 F + Dt = 0. \tag{6.16}$$

The values of the integration constants A to H in the general solution for $\mathbf{\Gamma}$ and Π given by (6.10), (6.11), (6.14) and (6.15) are determined by using the outer boundary conditions (6.2*b*) on $\tilde{\mathbf{u}}_1$ together with (6.16). The flow field $\tilde{\mathbf{u}}_1, \tilde{p}_1$ may then in principle be obtained by taking inverse Fourier transforms of $\mathbf{\Gamma}$ and Π.

7. Forces acting on a sphere in terms of Fourier transforms

In the following sections the cases of one or two spherical particles sedimenting through a stagnant fluid bounded by an infinite plane wall W are investigated. For these cases, the expressions for $\mathbf{\Gamma}$ are quite complex and it is not always possible to invert these Fourier transforms analytically. However, it will now be shown that the force $\mathbf{F}$ experienced by the particle may be obtained directly from $\mathbf{\Gamma}$.

For the sedimentation of a spherical particle in an unbounded fluid, the outer boundary conditions (6.2*b*) must be replaced by

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r} \to \infty, \tag{7.1}$$

the solution for $\tilde{\mathbf{u}}_1, \tilde{p}_1$ then being (Brenner & Cox 1963; Vasseur & Cox 1976)

$$\tilde{\mathbf{u}}_1 = \frac{3\tilde{\mathbf{r}}}{2\tilde{r}^3} - \frac{3}{4}\left\{\frac{\mathbf{e}_1}{\tilde{r}} + \frac{\tilde{\mathbf{r}}}{\tilde{r}^2}\left(1+\frac{2}{\tilde{r}}\right)\right\} e^{\frac{1}{2}(\tilde{r}_1 - \tilde{r})}, \quad \tilde{p}_1 = -\frac{3\tilde{r}_1}{2\tilde{r}^3}, \tag{7.2}$$

which for $\tilde{r} \to 0$ is of the form

$$\tilde{\mathbf{u}}_1 = -\frac{3}{4}\left\{\frac{\mathbf{e}_1}{\tilde{r}} + \frac{\tilde{r}_1 \tilde{\mathbf{r}}}{\tilde{r}^3}\right\} + \frac{3}{16}\left\{2\left(1-\frac{\tilde{r}_1}{\tilde{r}}\right)\mathbf{e}_1 + \left(\frac{\tilde{r}^2 - \tilde{r}_1^2}{\tilde{r}^3}\right)\tilde{\mathbf{r}}\right\} + O(\tilde{r}). \tag{7.3}$$

If a solid wall or other particles are present, the asymptotic form of $\tilde{\mathbf{u}}_1$ as $\tilde{r} \to 0$ must be given by (7.3) except that now a disturbance flow $\tilde{\mathbf{u}}_d$, due to the walls or other particles, must be added to it. Since such a disturbance flow is not singular as $\tilde{r} \to 0$, it may be expanded in a Taylor series as

$$\tilde{\mathbf{u}}_d = \mathbf{d} + O(\tilde{r}), \tag{7.4}$$

where $\mathbf{d}$ is the value of $\tilde{\mathbf{u}}_d$ at $\tilde{\mathbf{r}} = 0$. Thus as $\tilde{r} \to 0$,

$$\tilde{\mathbf{u}}_1 = -\frac{3}{4}\left(\frac{\mathbf{e}_1}{\tilde{r}} + \frac{\tilde{r}_1 \tilde{\mathbf{r}}}{\tilde{r}^3}\right) + \frac{3}{16}\left\{2\left(1-\frac{\tilde{r}_1}{\tilde{r}}\right)\mathbf{e}_1 + \left(\frac{\tilde{r}^2 - \tilde{r}_1^2}{\tilde{r}^3}\right)\tilde{\mathbf{r}}\right\} + \mathbf{d} + O(\tilde{r}), \tag{7.5}$$

which when compared with (5.7) yields for this case

$$\boldsymbol{\gamma} = \tfrac{3}{8}\mathbf{e}_1 + \mathbf{d}, \tag{7.6}$$

so that the force $\mathbf{F}$ on the particle at $\tilde{\mathbf{r}} = 0$ given by (5.6a) is

$$\mathbf{F} = 6\pi\{\mathbf{e}_1 + Re\,(\tfrac{3}{8}\mathbf{e}_1 + \mathbf{d}) + o(Re)\}. \tag{7.7}$$

Defining $\boldsymbol{\Gamma}_v$ as the two-dimensional Fourier transform of the velocity field in (7.2) which satisfies the Oseen equation for an isolated point force at $\tilde{\mathbf{r}} = 0$, we see that the Fourier transform of $\tilde{\mathbf{u}}_d$ is $\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_v$ so that

$$\begin{aligned}\mathbf{d} &= \lim_{\tilde{\mathbf{r}} \to 0} \tilde{\mathbf{u}}_d = \lim_{\tilde{\mathbf{r}} \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_v)\exp\left[i(k_1\tilde{r}_1 + k_2\tilde{r}_2)\right] dk_1\,dk_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_v)_{\tilde{r}_3 \to 0}\, dk_1\,dk_2.\end{aligned} \tag{7.8}$$

The value of $\boldsymbol{\Gamma}_v$ may be obtained in a manner similar to that for $\boldsymbol{\Gamma}$ as

$$(\Gamma_v)_1 = -(3/4\pi)\{ik_1 q^{-1} e^{-q|\tilde{r}_3|} - (ik_1 - 1)\,t^{-1} e^{-t|\tilde{r}_3|}\}, \tag{7.9a}$$

$$(\Gamma_v)_2 = -(3/4\pi)\{ik_2 q^{-1} e^{-q|\tilde{r}_3|} - ik_2\, t^{-1} e^{-t|\tilde{r}_3|}\}, \tag{7.9b}$$

$$(\Gamma_v)_3 = +(3/4\pi)\,\mathrm{sgn}\,(\tilde{r}_3)\{e^{-q|\tilde{r}_3|} - e^{-t|\tilde{r}_3|}\}. \tag{7.9c}$$

If a velocity field $\tilde{\mathbf{u}}_s$ is defined as the flow satisfying the creeping-motion equations (3.2a) in an unbounded fluid that results from a point force of $6\pi\mathbf{e}_1$ at $\tilde{\mathbf{r}} = 0$, i.e.

$$\tilde{\mathbf{u}}_s = -\tfrac{3}{4}\tilde{r}^{-1}(\mathbf{e}_1 + \tilde{r}^{-2}\tilde{r}_1\tilde{\mathbf{r}}), \tag{7.10}$$

then its two-dimensional Fourier transform $\boldsymbol{\Gamma}_s$ may be obtained (Vasseur & Cox 1976) as

$$(\Gamma_s)_1 = (3/8\pi)\{-2q^{-1} + k_1^2 q^{-3}(1 + q|\tilde{r}_3|)\} e^{-q|\tilde{r}_3|}, \tag{7.11a}$$

$$(\Gamma_s)_2 = (3/8\pi)\,k_1 k_2 q^{-3}(1 + q|\tilde{r}_3|)\,e^{-q|\tilde{r}_3|}, \tag{7.11b}$$

$$(\Gamma_s)_3 = (3/8\pi)\,ik_1 q^{-1}\tilde{r}_3\, e^{-q|\tilde{r}_3|}. \tag{7.11c}$$

It may then be shown by direct substitution that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{\Gamma}_v-\mathbf{\Gamma}_s)_{\tilde{r}_s\to 0}\,dk_1\,dk_2 = \tfrac{3}{8}\mathbf{e}_1. \tag{7.12}$$

Thus the value of $\mathbf{d}$ given by (7.8) may be written as

$$\mathbf{d} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{\Gamma}-\mathbf{\Gamma}_s)_{\tilde{r}_s\to 0}\,dk_1\,dk_2-\tfrac{3}{8}\mathbf{e}_1, \tag{7.13}$$

so that the force $\mathbf{F}$ on the particle given by (7.7) is

$$\mathbf{F} = 6\pi\left\{\mathbf{e}_1+Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{\Gamma}-\mathbf{\Gamma}_s)_{\tilde{r}_s\to 0}\,dk_1\,dk_2+o(Re)\right\}, \tag{7.14}$$

which when written in dimensional form becomes

$$\mathbf{F}' = 6\pi\mu a V\left\{\mathbf{e}_1+Re\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{\Gamma}-\mathbf{\Gamma}_s)_{\tilde{r}_s\to 0}\,dk_1\,dk_2+o(Re)\right\}. \tag{7.15}$$

8. Sphere sedimenting in a stagnant fluid bounded by a plane wall

In this section, the motion of a single sphere of radius a sedimenting with a velocity V through a viscous quiescent fluid bounded by an infinite vertical plane wall W at $r_3' = -d$ is considered, with co-ordinate axes (defined as in § 2) moving with the body so that the fluid velocity at infinity and on the surface of the wall W is $(V, 0, 0)$. The wall is assumed to be located at such a large distance from the particle that it lies within the outer region of expansion (i.e. $dV/\nu = O(1)$). Thus the outer boundary conditions on $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ are

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r}\to\infty, \quad \tilde{\mathbf{u}}_1 = 0 \quad \text{on} \quad W, \tag{8.1}$$

so that their Fourier transforms $\mathbf{\Gamma}$, Π must satisfy

$$\Gamma_1, \Gamma_2, \Gamma_3 \to 0 \quad \text{as} \quad \tilde{r}_3\to\infty, \tag{8.2a}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{on} \quad \tilde{r}_3 = -d^*, \tag{8.2b}$$

where $d^* = dV/\nu$ is the dimensionless distance between the centre of the sphere and the wall (d/a) expressed in terms of the outer co-ordinate system defined in (3.4). The unknown constants $A, B, \ldots, H$ appearing in the expressions (6.10), (6.14) and (6.15) for $\mathbf{\Gamma}$ may now be calculated for this case by substituting into (8.2) and making use of the continuity equations (6.16). The value of $\mathbf{\Gamma}$ is thus obtained as

$$\begin{aligned}\Gamma_1 = &-\frac{3ik_1(t+q)}{4\pi q(t-q)}\exp\left[-q(\tilde{r}_3+2d^*)\right]\\ &-\frac{3}{4\pi}\left\{\frac{2ik_1 t}{(t-q)\,q}+\frac{ik_1-1}{t}\right\}\exp\left[-t(\tilde{r}_3+2d^*)\right]\\ &+\frac{3ik_1 t}{2\pi(t-q)\,q}\{\exp\left[-q(\tilde{r}_3+d^*)-td^*\right]+\exp\left[-t(\tilde{r}_3+d^*)-qd^*\right]\}+(\Gamma_v)_1,\end{aligned} \tag{8.3a}$$

$$\Gamma_2 = -\frac{3(t+q)\,ik_2}{4\pi(t-q)\,q}\exp\left[-q(\tilde{r}_3+2d^*)\right]$$
$$-\frac{3}{4\pi}ik_2\left\{\frac{1}{t}+\frac{2t}{(t-q)\,q}\right\}\exp\left[-t(\tilde{r}_3+2d^*)\right]$$
$$+\frac{3ik_2t}{2\pi(t-q)\,q}\{\exp\left[-q(\tilde{r}_3+d^*)-qd^*\right]+\exp\left[-q(\tilde{r}_3+d^*)-td^*\right]\}+(\Gamma_v)_2, \tag{8.3b}$$

$$\Gamma_3 = \frac{3(t+q)}{4\pi(t-q)}\{\exp\left[-q(\tilde{r}_3+2d^*)\right]+\exp\left[-t(\tilde{r}_3+2d^*)\right]\}$$
$$-\frac{3}{2\pi}\left\{\frac{t}{t-q}\exp\left[-q(\tilde{r}_3+d^*)-td^*\right]+\frac{q}{t-q}\exp\left[-t(\tilde{r}_3+d^*)-qd^*\right]\right\}+(\Gamma_v)_3, \tag{8.3c}$$

where $(\Gamma_v)_1, (\Gamma_v)_2, (\Gamma_v)_3$ are given by (7.9).

Lift velocity

The lift force $\mathbf{F}'_l$ (i.e. the force orthogonal to the direction of motion of the particle) is given in the present case by the component of $\mathbf{F}'$ in the $\mathbf{e}_3$ direction. Furthermore, to the lowest non-zero order in the Reynolds number, the lift velocity v'_l may be obtained from $F'_l = 6\pi\mu a v'_l$. Thus from (7.15) one obtains

$$\mathbf{v}'_l = V\,Re\,I_3\,\mathbf{e}_3, \tag{8.4}$$

where

$$I_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\Gamma_3-(\Gamma_s)_3\}_{\tilde{r}_s\to 0}\,dk_1\,dk_2. \tag{8.5}$$

Substituting the value of $(\Gamma_s)_3$ from (7.11) and Γ_3 from (8.3c), we obtain

$$I_3 = \frac{3}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(k_1^2+k_2^2+ik_1)^{\frac{1}{2}}+(k_1^2+k_2^2)^{\frac{1}{2}}}{(k_1^2+k_2^2+ik_1)^{\frac{1}{2}}-(k_1^2+k_2^2)^{\frac{1}{2}}}\{\exp\left[-(k_1^2+k_2^2)^{\frac{1}{2}}d^*\right]$$
$$-\exp\left[-(k_1^2+k_2^2+ik_1)^{\frac{1}{2}}d^*\right]\}^2\,dk_1\,dk_2. \tag{8.6}$$

To evaluate this integral the following substitutions are made:

$$k_1 = \rho\cos\phi/d^*, \quad k_2 = \rho\sin\phi/d^*, \tag{8.7}$$

so that I_3 may be written as

$$I_3 = \frac{3}{4\pi d^{*2}}\int_0^{\infty}\int_0^{2\pi}\frac{(\rho^2+id^*\rho\cos\phi)^{\frac{1}{2}}+\rho}{(\rho^2+id^*\rho\cos\phi)^{\frac{1}{2}}-\rho}(\exp\left[-\rho\right]$$
$$-\exp\left[-\rho^2+id^*\rho\cos\phi)^{\frac{1}{2}}\right])^2\,\rho\,d\rho\,d\phi. \tag{8.8}$$

Although this integral cannot be evaluated analytically, its asymptotic form for small and large values of d^* may be obtained.

Thus if the distance d^* between the sphere and the wall is small (i.e. $d^* \leqslant 1$), one may use the following expansions:

$$(\rho^2+id^*\rho\cos\phi)^{\frac{1}{2}} = \rho+\frac{id^*\cos\phi}{2}+\frac{d^{*2}\cos^2\phi}{8\rho}-\frac{id^{*3}\cos^3\phi}{16\rho^2}-\frac{5d^{*4}\cos^4\phi}{128\rho^3}+\dots \tag{8.9}$$

and

$$\exp[(-\rho^2+id^*\rho\cos\phi)^{\frac{1}{2}}] = e^{-\rho}\left\{1-\frac{id^*\cos\phi}{2}-\frac{d^{*2}\cos^2\phi}{8\rho}(\rho+1)\right.$$

$$\left.+\frac{id^{*3}\cos^3\phi}{48\rho^2}(\rho^2+3\rho+3)+\frac{d^{*4}\cos^4\phi}{384\rho^3}(\rho^3+6\rho^2+15\rho+15)+\ldots\right\}. \quad (8.10)$$

Substituting (8.9) and (8.10) into (8.8) and performing the ρ and ϕ integration, one obtains

$$I_3 = \tfrac{3}{32}(1-\tfrac{11}{32}d^{*2}+\ldots). \quad (8.11)$$

It should be noted that (8.11) is obtained by assuming that the expansions (8.9) and (8.10) are valid. This occurs when $d^*\cos\phi/\rho < 1$ and thus when $d^* < \rho$. However, the contribution to I_3 from the region where the expansions are not valid (i.e. where $\rho < d^* \ll 1$) is negligible since the integrand in I_3 is finite as $\rho \to 0$. Substitution of (8.11) into (8.4) yields the lift velocity as

$$v_l' = \tfrac{3}{32}(aV^2/\nu)\{1-\tfrac{11}{32}(dV/\nu)^2+\ldots\}. \quad (8.12)$$

The first term in (8.12) was obtained by Cox & Hsu (1976) on the basis of a completely different method. This result is also in agreement with the result obtained by Oseen (1927) on the basis of the Oseen equations.

Evaluation of I_3 will now be considered for the limiting case where the distance d^* between the sphere and the wall is large (i.e. $d^* \gg 1$). For convenience (8.8) may be rewritten in the form

$$I_3 = \frac{3}{4\pi d^{*2}}\int_0^\infty\int_0^{2\pi}\frac{(i\rho\cos\phi+\rho^2/d^*)^{\frac{1}{2}}+\rho/d^{*\frac{1}{2}}}{(i\rho\cos\phi+\rho^2/d^*)^{\frac{1}{2}}-\rho/d^{*\frac{1}{2}}}(\exp[-\rho]$$

$$-\exp[(i\rho\cos\phi+\rho^2/d^*)^{\frac{1}{2}}d^{*\frac{1}{2}}])^2\rho\,d\rho\,d\phi, \quad (8.13)$$

where for large d^*

$$(i\rho\cos\phi+\rho^2/d^*)^{\frac{1}{2}} = (i\rho\cos\phi)^{\frac{1}{2}}\{1+\tfrac{1}{2}(\rho/id^*\cos\phi)-\tfrac{1}{8}(\rho/id^*\cos\phi)^2$$

$$+\tfrac{1}{16}(\rho/id^*\cos\phi)^3+\ldots\}. \quad (8.14)$$

Substituting this approximation into (8.13) and performing the resulting ρ and ϕ integrations, one obtains

$$I_3 = \tfrac{3}{8}(d^*)^{-2}+\tfrac{9}{8}(2\pi)^{-\frac{1}{2}}K(\tfrac{1}{2})(d^*)^{-\frac{5}{2}}+\ldots, \quad (8.15)$$

where $K(m)$ is the complete elliptic integral of the first kind. Substituting this result into (10.4) one obtains for the lift velocity

$$v_l' = \tfrac{3}{8}(aV^2/\nu)\{(\nu/dV)^2+\alpha(\nu/dV)^{\frac{5}{2}}+\ldots\}, \quad (8.16)$$

where

$$\alpha = 3K(\tfrac{1}{2})/(2\pi)^{\frac{1}{2}} = 2{\cdot}21901. \quad (8.17)$$

It is interesting to note that the leading term in this expression for the lift velocity v_l' experienced by a sphere sedimenting at a large distance from a wall does not depend on the sedimentation velocity V.

In order to obtain a solution for the lift velocity valid fcr the entire range of values of d^*, a numerical integration of (8.8) was undertaken. The results are presented in figure 2, where the migration velocity, normalized by $aV^2/\nu = V\,Re$, is plotted as a function of the variable $d^*(=dV/\nu)$. A positive value of v_l' was

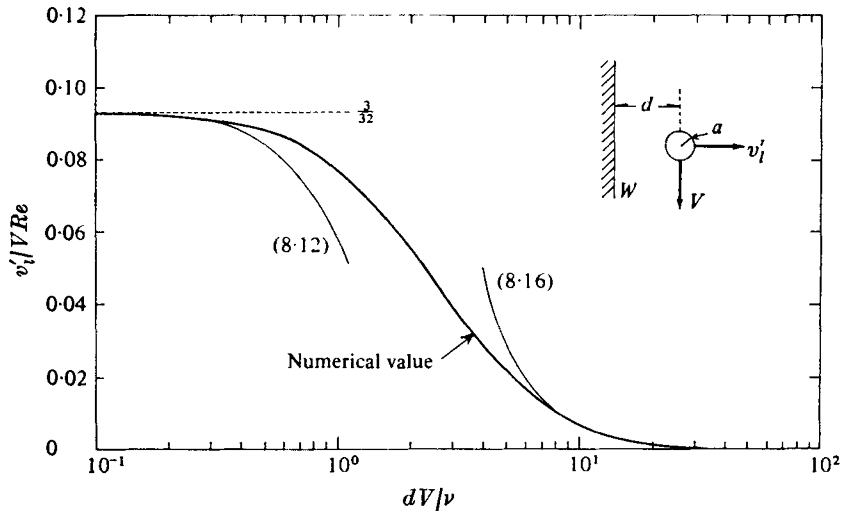

FIGURE 2. Lift velocity experienced by a sphere sedimenting in a fluid bounded by a plane wall.

obtained for all d^*, indicating that the sphere always moves away from the wall. It is seen that for values of $d^* < 0{\cdot}2$ (i.e. when the particle is in the vicinity of the wall), the sphere migrates with a constant velocity, namely $v'_l/V\,Re = 3/32$, as predicted by (8.12). For larger values of d^*, the migration velocity decreases continuously and tends to zero as $d^* \to \infty$. The results given by (8.12) and (8.16) are also included in this graph for comparison.

Drag force

The drag force F'_d experienced by the particle is given, in the present problem, by the component of $\mathbf{F}'$ in the $\mathbf{e}_1$ direction. Thus, from (7.15), one obtains

$$F'_d = 6\pi\mu a V\{1 + Re\, I_1 + \ldots\}, \tag{8.18}$$

where

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\Gamma_1 - (\Gamma_s)_1\}_{\tilde{r}_3\to 0}\, dk_1\, dk_2. \tag{8.19}$$

Substituting the value of Γ_1 from (8.3a) into (8.19) and evaluating the resulting integrand for $\tilde{r}_3 \to 0$, one obtains

$$I_1 = -\frac{3}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{ik_1(t+q)}{q(t-q)}e^{-2qd^*} + \left\{\frac{2ik_1 t}{(t-q)q} + \frac{ik_1 - 1}{t}\right\}e^{-2td^*} - \frac{4ik_1 t}{(t-q)q}e^{-(t+q)d^*}\right]dk_1\, dk_2$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\Gamma_v)_1 - (\Gamma_s)_1\}_{\tilde{r}_3\to 0}\, dk_1\, dk_2, \tag{8.20}$$

where $(\Gamma_v)_1$ and $(\Gamma_s)_1$ are defined by (7.9) and (7.11) respectively.

The value of the second integral in the right-hand side of (8.20) is given by (9.12) as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\Gamma_v)_1 - (\Gamma_s)_1\}_{\tilde{r}_3\to 0}\, dk_1\, dk_2 = \tfrac{9}{8}. \tag{8.21}$$

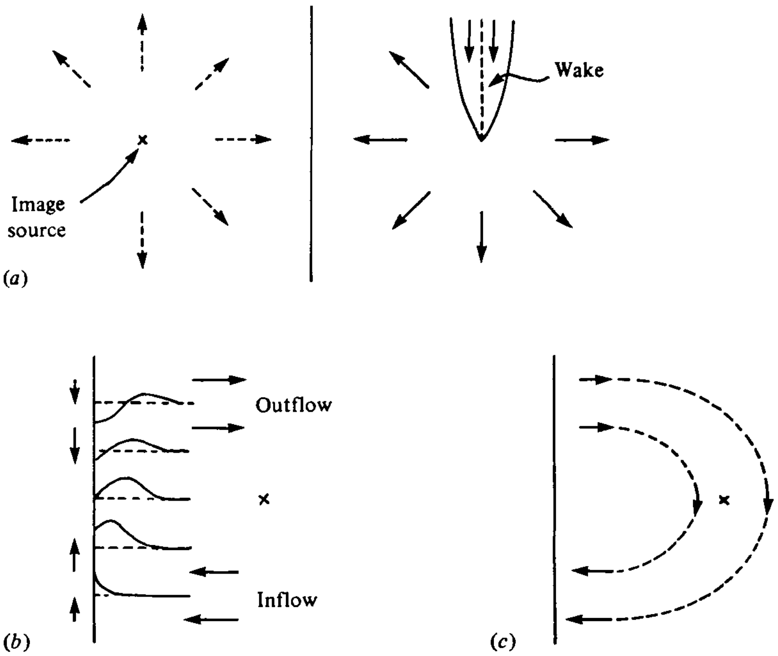

FIGURE 3. Disturbance flows for large d^*. (a) Flow due to particle and its image in vertical wall. (b) Boundary layer on the wall showing related inflow and outflow resulting from volume flux changes. (c) Induced potential flow due to inflow and outflow from boundary layer. This flow causes a drag reduction on the particle.

Introducing the parameters defined by (8.7) into (8.20) and making use of (8.21) one obtains

$$I_1 = \tfrac{3}{8} - \frac{3}{4\pi d^{*2}}\int_0^\infty\int_0^{2\pi}\left[(\chi+\rho)\,e^{-2\rho} + \left\{2\chi + \frac{\chi-\rho}{\chi}\,\frac{(i\rho\cos\phi - d^*)}{i\rho\cos\phi}\right\}e^{-2\chi} - 4\chi\, e^{-(\chi+\rho)}\right]\frac{i\rho\cos\phi}{\chi-\rho}\,d\rho\,d\phi, \quad (8.22)$$

where
$$\chi = (\rho^2 + i\rho d^*\cos\phi)^{\frac{1}{2}}. \quad (8.23)$$

The asymptotic forms of I_1 for small and for large values of d^* may be found in a manner similar to that used for I_3. Thus one may obtain

$$I_1 \sim \tfrac{9}{16}d^{*-1} + \tfrac{3}{8} + \dots \quad \text{as} \quad d^* \to 0, \quad (8.24)$$

and
$$I_1 \sim \tfrac{3}{8}\{1 - \beta d^{*-\frac{5}{2}} + \dots\} \quad \text{as} \quad d^* \to \infty, \quad (8.25)$$

where
$$\beta = 3(2\pi)^{-\frac{1}{2}}\{E(\tfrac{1}{2}) - \tfrac{1}{2}K(\tfrac{1}{2})\} = 0{\cdot}50698.$$

Substituting (8.24) into (8.18), it is seen that the drag force F'_d experienced by a sphere sedimenting in the neighbourhood of a plane wall ($a \ll d \ll \nu/V$) is

$$F'_d = 6\pi\mu a V\{1 + \tfrac{3}{8}Re + \tfrac{9}{16}a/d + \dots\}. \quad (8.26)$$

In the limit $Re \to 0$, this reduces to

$$F'_d = 6\pi\mu a V\{1 + \tfrac{9}{16}a/d + \dots\}, \quad (8.27)$$

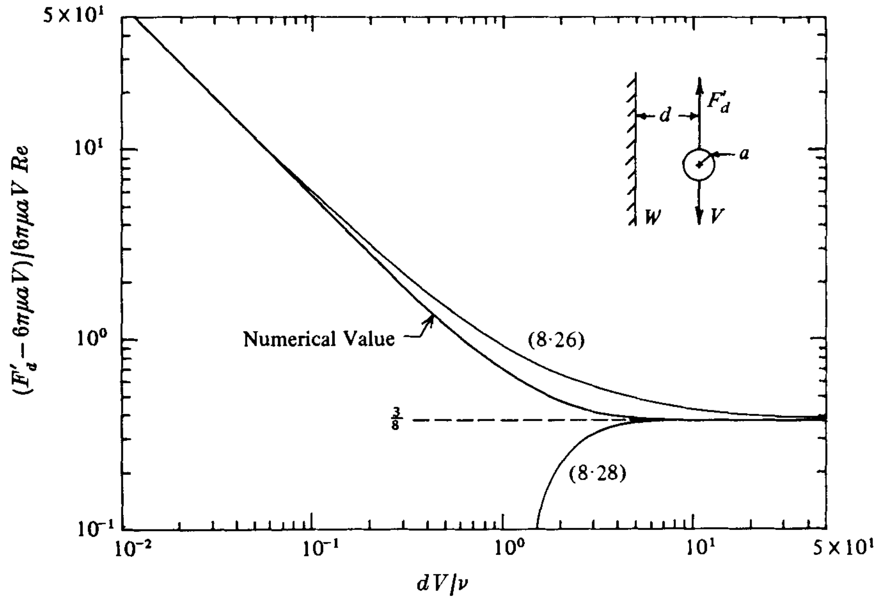

FIGURE 4. Drag force experienced by a sphere sedimenting in a fluid bounded by a plane wall.

which agrees with the values obtained by Lorentz (1907) and by Faxén (1922) on the basis of the creeping-motion equations (Happel & Brenner 1965). Substituting (8.25) into (8.18), one obtains the drag force on a sphere at a large distance ($a \ll \nu/V \ll d$) from the plane wall as

$$F'_d = 6\pi\mu a V\{1 + \tfrac{3}{8}Re - \tfrac{3}{8}\beta(a/d)(\nu/dV)^{\frac{3}{2}} + \ldots\}. \tag{8.28}$$

It is interesting to note that, unlike the situation when d^* is small, the presence of the wall causes a reduction in the drag force on the sphere. This surprising result may be explained by noting that the inflow and outflow of fluid from the boundary layer over the wall can induce a potential flow which close to the sphere is locally in the direction of the sphere's motion (see figure 3).

The numerical solution of the integral in (8.22) has been carried out and the result is presented in figure 4, where for comparsion the values for small and large d^* given by (8.26) and (8.28) have been included.

9. Interaction between two spherical particles

In this section the motion of two spherical particles sedimenting at small Reynolds numbers in a quiescent unbounded viscous fluid is considered. The two spherical particles of radii a and b (which will be referred to as spheres a and b respectively) are assumed to move with instantaneous velocities V and V_b downwards and are allowed to rotate as they fall with zero torque acting on them. Then their angular velocities $\boldsymbol{\omega}'_a$ and $\boldsymbol{\omega}'_b$ would be $o(Re)$ as was discussed in §5. As in §2 Cartesian co-ordinate axes (r'_1, r'_2, r'_3), with origin at the centre of sphere a, are chosen such that the r'_1 direction is vertically upwards. The position of the centre of sphere b with respect to these axes is taken to be at $(r_1', r_2', r_3') = (s, h, c)$. The distance l between the particle centres is assumed to be very much larger than the particle radius a (i.e. $a/l \ll 1$) and the particles are supposed to be sufficiently separated so that sphere b is located in the outer region of expansion of sphere a (i.e. $lV/\nu = O(1)$). In this co-ordinate system the flow is steady provided that the inequality $(V - V_b)/V \ll 1$ is satisfied (Vasseur 1973) and the fluid at infinity has the velocity $(V, 0, 0)$. The velocity field $\tilde{\mathbf{u}}_1, \tilde{p}_1$ for the present problem satisfies the Oseen equations and matches onto inner expansions at both sphere a and sphere b. The matching of this first-order outer expansion onto the inner expansion at sphere a requires that $\tilde{\mathbf{u}}_1, \tilde{p}_1$ behave as $\tilde{r} \to 0$ as though a point force acts on the fluid equal in magnitude to the drag on sphere a. Similarly matching onto the inner expansion at sphere b requires that $\tilde{\mathbf{u}}_1, \tilde{p}_1$ behave in the vicinity of sphere b as though a point force acts equal to the force sphere b exerts on the fluid.

Thus, for the present problem $\tilde{\mathbf{u}}_1, \tilde{p}_1$ satisfy

$$\tilde{\nabla}^2 \tilde{\mathbf{u}}_1 - \tilde{\nabla} \tilde{p}_1 - \partial \tilde{\mathbf{u}}_1 / \partial \tilde{r}_1 = 6\pi \mathbf{e}_1 \{\delta(\tilde{\mathbf{r}}) + K\delta(\tilde{r}_1 - s^*)\,\delta(\tilde{r}_2 - h^*)\,\delta(\tilde{r}_3 - c^*)\}, \quad (9.1)$$

$$\tilde{\nabla} . \tilde{\mathbf{u}}_1 = 0, \quad (9.2)$$

together with the boundary condition

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r} \to \infty, \quad (9.3)$$

where

$$s^* = sV/\nu, \quad h^* = hV/\nu, \quad c^* = cV/\nu, \quad (9.4)$$

and where

$$K = V_b b / V_a a. \quad (9.5)$$

Thus the Fourier transforms $\mathbf{\Gamma}$, Π of $\tilde{\mathbf{u}}_1, \tilde{p}_1$ satisfy

$$\left\{ -k_1^2 - k_2^2 + \frac{\partial^2}{\partial \tilde{r}_3^2} \right\} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} - \begin{pmatrix} ik_1 \\ ik_2 \\ \partial/\partial \tilde{r}_3 \end{pmatrix} \Pi - ik_1 \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} = \frac{1}{4\pi^2} \begin{pmatrix} 6\pi \\ 0 \\ 0 \end{pmatrix} \times \{\delta(\tilde{r}_3) + K e^{\Lambda} \delta(\tilde{r}_3 - c^*)\}, \quad (9.6)$$

$$i(k_1 \Gamma_1 + k_2 \Gamma_2) + \partial \Gamma_3 / \partial \tilde{r}_3 = 0, \quad (9.7)$$

where

$$\Lambda = -i(k_1 s^* + k_2 h^*), \quad (9.8)$$

together with

$$\mathbf{\Gamma} \to 0 \quad \text{as} \quad \tilde{r}_3 \to \pm \infty. \quad (9.9)$$

By a suitable choice of axes (so that the r_1, r_3 plane contains the spheres) h^* may be taken to be zero. Then (9.6)–(9.9) may be solved in a manner similar to that for the case considered in § 8, as

$$\Gamma_1 = -(3/4\pi)\{ik_1 q^{-1}(a_1 + K a_2 e^{\xi}) + (ik_1 - 1)\, t^{-1}(a_3 + K a_4 e^{\xi})\}, \quad (9.10)$$

$$\Gamma_2 = -(3/4\pi)\, ik_2 \{q^{-1}(a_1 + K a_2 e^{\xi}) + t^{-1}(a_3 + K a_4 e^{\xi})\}, \quad (9.11)$$

$$\Gamma_3 = (3/4\pi)\{\operatorname{sgn}(\tilde{r}_3)\,(a_1 + a_3) + \operatorname{sgn}(\tilde{r}_3 - c^*)\, K(a_2 + a_4)\, e^{\xi}\}, \quad (9.12)$$

$$\Pi = (3/4\pi)\, ik_1 q^{-1}\{a_1 + K a_2 e^{\xi}\}, \quad (9.13)$$

where

$$= -ik_1 s^*, \quad a_1 = e^{-q|\tilde{r}_3|}, \quad a_2 = e^{-q|\tilde{r}_3 - c^*|}, \quad a_3 = e^{-t|\tilde{r}_3|}, \quad a_4 = e^{-t|\tilde{r}_3 - c^*|}. \quad (9.14), (9.15)$$

Substituting (9.10)–(9.13) into (7.2) and performing the resulting integrals one obtains

$$\tilde{\mathbf{u}}_1 = -\tfrac{3}{4}\{\tilde{r}^{-1}e^{\frac{1}{2}(\tilde{r}_1-\tilde{r})}(\mathbf{e}_1+a_5\tilde{\mathbf{r}})+\tilde{R}^{-1}e^{\frac{1}{2}(\tilde{R}_1-\tilde{R})}K(\mathbf{e}_1+a_6\tilde{\mathbf{R}})\}+\tfrac{3}{2}\{\tilde{r}^{-3}\tilde{\mathbf{r}}+K\tilde{R}^{-3}\tilde{\mathbf{R}}\}, \tag{9.16}$$

$$\tilde{p}_1 = -\tfrac{3}{2}\{\tilde{r}_1\tilde{r}^{-3}+K\tilde{R}_1\tilde{R}^{-3}\}, \tag{9.17}$$

where

$$\tilde{\mathbf{R}} = (\tilde{r}_1-s^*)\,\mathbf{e}_1+\tilde{r}_2\,\mathbf{e}_2+(\tilde{r}_3-c^*)\,\mathbf{e}_3, \quad \tilde{R} = |\tilde{\mathbf{R}}|, \tag{9.18a}$$

$$a_5 = \tilde{r}^{-2}(\tilde{r}+2), \quad a_6 = \tilde{R}^{-2}(\tilde{R}+2). \tag{9.18b}$$

By expanding the expression for $\tilde{\mathbf{u}}_1$, as given by (9.17), about $\tilde{r} = 0$, one obtains

$$\begin{aligned}\tilde{\mathbf{u}}_1 = & -\tfrac{3}{4}\tilde{r}^{-1}(\mathbf{e}_1+\tilde{r}_1\tilde{\mathbf{r}}/\tilde{r}^2)+\tfrac{3}{16}\{2(1-\tilde{r}_1/\tilde{r})\,\mathbf{e}_1+(1-\tilde{r}_1^2/\tilde{r}^2)\,\tilde{\mathbf{r}}/\tilde{r}\}\\ & -(3K/4l^{*2})\sin\theta\{2-(l^*+2-l^*/\sin\theta)\exp[-\tfrac{1}{2}l^*(\sin\theta+1)]\}\,\mathbf{e}_1\\ & -(3K/2l^{*2})\cos\theta\{1-(\tfrac{1}{2}l^*+1)\exp[-\tfrac{1}{2}l^*(\sin\theta+1)]\}\,\mathbf{e}_3+\ldots,\end{aligned} \tag{9.19}$$

where $l^* = lV/\nu$ is the dimensionless distance (l/a) between the sphere centres expressed in terms of the stretched variables and θ is the angle made by the line joining the sphere centres with the horizontal. Thus

$$\sin\theta = s^*/l^*, \quad \cos\theta = c^*/l^*, \quad l^* = (c^{*2}+s^{*2})^{\frac{1}{2}}. \qquad (9.20),\ (9.21)$$

By comparing (9.19) with (7.5) the value of $\mathbf{d}$ may be obtained, which when substituted in (7.7) yields the drag force F_d' on sphere a as

$$\begin{aligned}F_d' = 6\pi\mu aV[1+\tfrac{3}{8}\,Re-\tfrac{3}{4}(a/ll^*)\,K\sin\theta\{2-(l^*+2-l^*/\sin\theta)\\ \times\exp[-\tfrac{1}{2}l^*(\sin\theta+1)]\}+\ldots],\end{aligned} \tag{9.22}$$

and the lift force F_l' as

$$F_l' = -\tfrac{9}{2}\pi\mu aV(a/ll^*)\,K\cos\theta\{2-(l^*+2)\exp[-\tfrac{1}{2}l^*(\sin\theta+1)]\}+\ldots. \tag{9.23}$$

The lift velocity v_l' is obtained from (9.23) by means of Stokes' law:

$$v_l' = F_l'/6\pi\mu a. \tag{9.24}$$

From these results it is observed that for two equal-sized spheres ($K = 1$):

(i) If they are sedimenting in the same horizontal plane ($\theta = 0$), the lift velocity of sphere a is

$$v_l' = -\tfrac{3}{4}(a/l)\;V(l^*)^{-1}\{2-(l^*+2)\exp(-\tfrac{1}{2}l^*)\}, \tag{9.25}$$

which is negative for all l^*, so that the spheres are repelled from each other. Furthermore the drag force on each sphere is

$$F_d' = 6\pi\mu aV\{1+\tfrac{3}{8}\,Re-\tfrac{3}{4}(a/l)\exp(-\tfrac{1}{2}l^*)\}, \tag{9.26}$$

so that they sediment faster than an isolated sphere of the same size.

(ii) If they are sedimenting vertically one above another, the drag force on the leading sphere is

$$F_d' = 6\pi\mu aV\{1+\tfrac{3}{8}Re-\tfrac{3}{2}(a/ll^*)\,(1-e^{-l^*})\}, \tag{9.27}$$

and on the trailing sphere

$$F_d' = 6\pi\mu aV\{1+\tfrac{3}{8}Re-\tfrac{3}{2}a/l\}, \tag{9.28}$$

so that each sphere sediments faster than an isolated sphere. Furthermore, the velocity of the trailing sphere exceeds that of the leading sphere by

$$\tfrac{3}{2}(a/l)\, V_\infty\, (l^*)^{-1}(e^{-l^*} - 1 - l^*),$$

where V_∞ is the sedimentation velocity of an isolated sphere. Since this is positive for all l^*, it follows that the trailing sphere always catches up the leading sphere.

(iii) The above results (9.25)–(9.28) agree with those obtained by Oseen (1927) except that now the term $\tfrac{3}{8}\,Re$ appears in the expressions for the drag force. Furthermore the values of F'_d agree with that obtained by Proudman & Pearson (1957) in the limit of $l^* \to \infty$.

(iv) For general positions of the spherical particles the rate of separation of the particles in the $\mathbf{e}_3$ direction is

$$\tfrac{3}{4}(a/l)\; V\cos\theta(l^*)^{-1}\{4 - (l^*+2)\, e^{-\frac{1}{2}l^*}2\cosh(\tfrac{1}{2}l^*\sin\theta)\}.$$

This is positive and thus represents a repulsion between the particles for (*a*) $l^* \to \infty$ with θ fixed ($\neq \pm\tfrac{1}{2}\pi$) and (*b*) $l^* \to 0$ for all θ. However it is negative for intermediate values of l^* near $\theta = \pm\tfrac{1}{2}\pi$ owing to fluid flux into the wake behind particles.

(v) The mean migration velocity of the particles in the $\mathbf{e}_3$ direction is

$$-\tfrac{3}{4}(a/l)\; V\cos\theta(l^*)^{-1}\,(l^*+2)\, e^{-l^*/2}\sinh(\tfrac{1}{2}l^*\sin\theta),$$

which is negative for $\theta > 0$ and positive for $\theta < 0$. This represents a migration in the same direction as that experienced at zero Reynolds number (Goldman, Cox & Brenner 1966).

(vi) The mean drag force on the spheres is

$$6\pi\mu a V[1 + \tfrac{3}{8}\,Re - \tfrac{3}{4}(a/l)\{(1+2/l^*)\sin\theta\sinh(\tfrac{1}{2}l^*\sin\theta) + \cosh(\tfrac{1}{2}l^*\cos\theta)\}\, e^{-l^*/2}$$

so that in all relative positions the mean drag force is reduced, showing that the mean sedimentation rate is always greater than that for an isolated sphere.

10. Two spheres sedimenting in a fluid bounded by an infinite plane wall

In this section the motion of two unequal spherical particles settling at small Reynolds numbers through a viscous quiescent fluid bounded by a vertical infinite plane wall W is considered.

Thus we have the situation considered in §9 except that now the fluid is bounded by a wall at $r'_3 = -d$ say. Thus the first-order outer flow field $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ satisfies the Oseen equations (9.1) and (9.2) with the boundary conditions

$$\tilde{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \tilde{r}_3 \to \infty, \quad \tilde{\mathbf{u}}_1 = 0 \quad \text{on} \quad \tilde{r}_3 = -d^*, \tag{10.1}$$

where

$$d^* = dV/\nu.$$

The Fourier transforms $\mathbf{\Gamma}$, Π of $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ then satisfy (9.6)–(9.8) with the boundary conditions

$$\Gamma_1, \Gamma_2, \Gamma_3 \to 0 \quad \text{as} \quad \tilde{r}_3 \to \infty, \tag{10.2}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{on} \quad \tilde{r}_3 = -d^*. \tag{10.3}$$

These equations may be solved in a manner similar to that used in § 8 to obtain

$$\begin{aligned}\tfrac{4}{3}\pi\Gamma_3 = {} & (t+q)(t-q)^{-1}\{b_1 e^{-q(\tilde{r}_3+2d^*)} + b_2 e^{-t(\tilde{r}_3+2d^*)}\} \\ & -2(t-q)^{-1} e^{-(t+q)d^*}\{q b_1 e^{-t\tilde{r}_3} + t b_2 e^{-q\tilde{r}_3}\} \\ & + \operatorname{sgn}\tilde{r}_3 (e^{-q|\tilde{r}_3|} - e^{-t|\tilde{r}_3|}) \\ & + \operatorname{sgn}(\tilde{r}_3 - c^*)(e^{-q|\tilde{r}_3-c^*|} - e^{-t|\tilde{r}_3-c^*|}) e^{\Lambda},\end{aligned} \tag{10.4}$$

where $$b_1 = 1 + K\exp(-qc^* + \Lambda), \quad b_2 = 1 + K\exp(-tc^* + \Lambda) \tag{10.5}$$

and Λ is given by (9.8). Substituting (10.4) and (7.11) into (8.5), one obtains the value of I_3, which when substituted into (8.4) yields

$$\begin{aligned} v_l' = \frac{3}{4\pi} V Re \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Bigg\{ & \frac{t+q}{t-q}(e^{-qd^*} - e^{-td^*})^2 \\ & - K\left[\frac{2e^{-(t+q)d^*}}{t-q}(t e^{-tc^*} + q e^{-qc^*}) - \frac{t+q}{t-q}\{e^{-q(c^*+2d^*)} + e^{-t(c^*+2d^*)}\}\right. \\ & \left. + \operatorname{sgn}(c^*)(e^{-q|c^*|} - e^{-t|c^*|})\right] e^{-i(k_1 s^* + k_2 h^*)}\Bigg\} dk_1\, dk_2. \end{aligned} \tag{10.6}$$

It should be noted that we are referring here only to the component of the lift in the $\mathbf{e}_3$ direction. Cases in which the spheres are aligned along the co-ordinate axes will now be investigated and it will be seen that for certain limiting cases (10.6) can be integrated analytically.

Motion of two spheres with their line of centres along the r_3' axis

The lift velocity v_l' produced by the fluid on sphere a may be obtained, for this situation, by letting $s^* = h^* = 0$ in (10.6). This yields

$$v_l' = V Re\,(I_{t1} + I_{t2} + I_{t3}], \tag{10.7}$$

where $$I_{t1} = \frac{3}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{t+q}{t-q}(e^{-qd^*} - e^{-td^*})\, dk_1\, dk_2, \tag{10.8}$$

$$I_{t2} = \frac{3K}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{t+q}{t-q} e^{-q(2d^*+c^*)} - \frac{2q}{t-q} e^{-q(d^*+c^*)-td^*} - (\operatorname{sgn} c^*) e^{-q|c^*|}\right\} dk_1\, dk_2, \tag{10.9}$$

$$I_{t3} = \frac{3K}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{t+q}{t-q} e^{-t(2d^*+c^*)} - \frac{2t}{t-q} e^{-t(d^*+c^*)-qd^*} + (\operatorname{sgn} c^*) e^{-t|c^*|}\right\} dk_1\, dk_2. \tag{10.10}$$

The integrals in (7.8)–(7.10) may be evaluated by numerical methods. However various asymptotic forms of (10.7) may be obtained in a manner similar to that discussed in § 8 for (8.6) and (8.20). Thus for sphere a located close to the wall W while sphere b is at a large distance from W (i.e. for $d^* \to 0$, $c^* \to \infty$), the lift velocity v_l' of sphere a is

$$v_l' = \tfrac{3}{32} V Re\{1 - \tfrac{11}{32}(dV/\nu)^2\} - Ka\nu(d^2/c^4)\{\alpha_1(cV/\nu)^{\frac{1}{2}} + 9\}, \tag{10.11}$$

where $$\alpha_1 = 45\{E(\tfrac{1}{2}) - \tfrac{1}{2}K(\tfrac{1}{2})\}/4\sqrt{\pi} = 2{\cdot}68869. \tag{10.12}$$

The first term in (10.11) represents the lift force on sphere a in the absence of sphere b (see (8.12)) while the second represents the small correction due to the presence of the sphere b.

Similarly it may be shown that, if the two particles a and b are interchanged, that is sphere b is in the vicinity of the wall W while sphere a is located at a large distance from it, so that $d^* \to \infty$ and $c^* + d^* \to 0$, then the migration velocity experienced by sphere a is

$$v_l' = (a\nu/d^2)\,[\tfrac{3}{8}\{1+\alpha_2(\nu/dV)^{\frac{1}{2}}\} + Kd^{-1}(c+d)\{\alpha_3(dV/\nu)^{\frac{1}{2}}+6\}], \qquad (10.13)$$

where

$$\alpha_2 = 3K(\tfrac{1}{2})/\sqrt{(2\pi)} = 2{\cdot}21901, \quad \alpha_3 = 9\{E(\tfrac{1}{2}) - \tfrac{1}{2}K(\tfrac{1}{2})\}/\sqrt{\pi} = 2{\cdot}15095. \qquad (10.14)$$

The second term in (10.13) represents the effect of the sphere b on the migration of sphere a. This is seen to be a small effect (since $c^* + d^* \to 0$ and $d^* \to \infty$) which tends to zero as the sphere b approaches the plane wall W.

Defining $\lambda_c = c^*/d^*$, other asymptotic forms for the lift velocity v_l' may be obtained similarly as follows:

(i) Both spheres close to wall (c^*, $d^* \to 0$):

$$v_l' = VRe\{\tfrac{3}{32} - \tfrac{3}{8}K(\lambda_c+2)^{-2}\} \quad \text{for left-hand sphere } (\lambda_c > 0), \qquad (10.15a)$$

$$= VRe\{\tfrac{3}{32} + \tfrac{3}{8}K(\lambda_c+1)(\lambda_c+3)/(\lambda_c+2)^2\} \quad \text{for right-hand sphere } (\lambda_c < 0), \qquad (10.15b)$$

the mean sphere lift velocity v_{lm}' being

$$v_{lm}' = VRe\{\tfrac{3}{32} + \tfrac{3}{8}K(\lambda_c+1)/(\lambda_c+2)^2\}, \qquad (10.15c)$$

where, in (10.15c), d^* has been taken as the dimensionless distance from the wall to the nearer sphere.

(ii) Spheres well separated from wall and from each other (c^*, $d^* \to \infty$):

$$v_l' = (a\nu/d^2)\{\tfrac{3}{8} - 6K(\lambda_c+1)/\lambda_c^2(\lambda_c+2)^2 \quad \text{for left-hand sphere } (\lambda_c > 0) \qquad (10.16a)$$

$$= (a\nu/d^2)\{\tfrac{3}{8} + 3K(\lambda_c^2+2\lambda_c+2)/\lambda_c^2(\lambda_c+2)^2\} \quad \text{for right-hand sphere } (\lambda_c < 0) \qquad (10.16b)$$

with

$$v_{lm}' = (a\nu/d^2)\{\tfrac{3}{16} + 3/16(\lambda_c+1)^2 + 3K/2(\lambda_c+2)^2\}. \qquad (10.16c)$$

(iii) Spheres close to each other but at large distance from wall ($c^* \to 0$, $d^* \to \infty$):

$$v_l' = \tfrac{3}{8}(a\nu/d^2)[1+K\{1-\tfrac{1}{2}(dV/\nu)^2\}] \quad \text{for left-hand sphere} \qquad (10.17a)$$

$$= \tfrac{3}{8}(a\nu/d^2)[1+K\{1+\tfrac{1}{2}(dV/\nu)^2\}] \quad \text{for right-hand sphere} \qquad (10.17b)$$

with

$$v_{lm}' = \tfrac{3}{8}(a\nu/d^2)(1+K). \qquad (10.17c)$$

Thus when the two particles are close to each other so that c^* is small the effect of the second particle is to increase the mean migration of the first particle by a factor of $(1+K)$. Thus the effect of a second particle is considerable, the mean migration velocity of a pair of particles being twice that of an isolated particle for two similar spheres ($K = 1$) when c^* is small. This will occur even when the

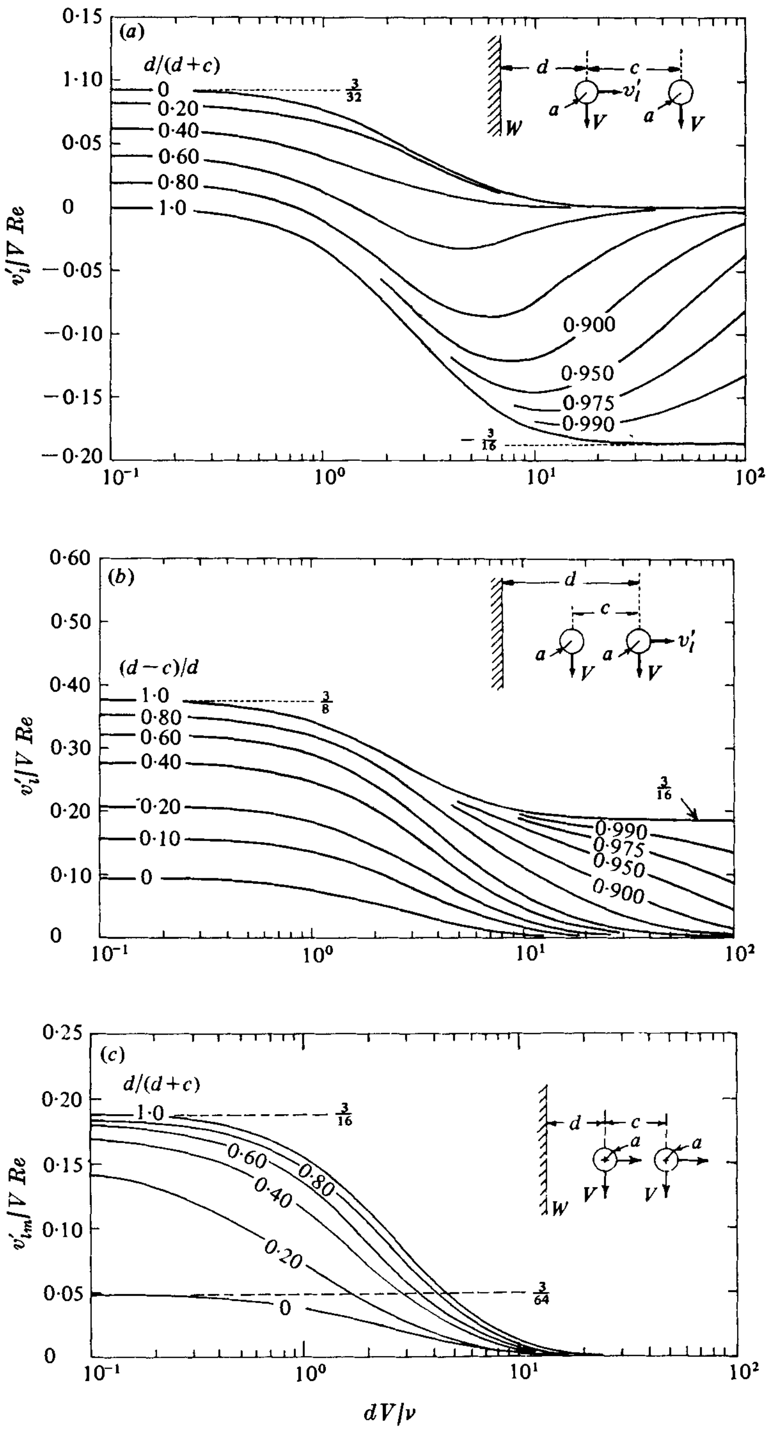

FIGURE 5. Lift velocity experienced by (a) the left-hand sphere and (b) the right-hand sphere of a pair of spheres sedimenting with their line of centres along the r_3' axis in the presence of a plane wall. (c) Their mean lift velocity.

separation distance c^* between the particles is very much larger than the particle size.

The migration velocity v_l' given by (10.7)–(10.10) was evaluated numerically for a pair of similar spheres (i.e. $K = 1$), the results being presented in figures 5(*a*) and (*b*) in which the migration velocity v_l' normalized with respect to VRe is plotted as a function of d^* for different ratios of the distance between the wall and the reference sphere and the distance between the sphere centres. Figure 5(*c*) gives in a similar fashion the values of the mean migration velocity v_{lm}'. These results show agreement with the asymptotic results given by (10.15), (10.16) and (10.17). It is to be noted how large an effect particle interaction has on migration velocity. For example, the mean migration velocity (see (10.15*c*) and figure 7) for $\lambda_c = 4$ $(d/(d+c) = 0{\cdot}2)$ is 55·6 % higher (for small d^*) than for an isolated particle.

Motion of two spheres with their line of centres along the r_2' axis

The lift velocity experienced by sphere a can be obtained, for this situation, by putting $s^* = c^* = 0$ in (10.6), which then reduces to

$$v_l' = \frac{3}{4\pi} VRe \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{t+q}{t-q} (e^{-qd^*} - e^{-td^*})^2 (1 + K e^{-ik_2 h^*})\, dk_1\, dk_2. \quad (10.18)$$

Letting $\lambda_h = h^*/d^*$, the various asymptotic forms of this result may be obtained as follows:

(i) Both spheres close to wall $(h^*, d^* \to 0)$:

$$v_l' = VRe\{\tfrac{3}{32} + 3K/4(4+\lambda_h^2)^{\frac{3}{2}}\} \quad \text{for either sphere.} \quad (10.19)$$

(ii) Spheres well separated from wall and from each other $(h^*, d^* \to \infty)$:

$$v_l' = (a\nu/d^2)\{\tfrac{3}{8} + 3K/(4+\lambda_h^2)^{\frac{3}{2}}\} \quad \text{for either sphere.} \quad (10.20)$$

(iii) Spheres close to each other but at large distance from wall $(h^* \to 0$, $d^* \to \infty)$:

$$v_l' = \tfrac{3}{8}(a\nu/d^2)(1+K) \quad \text{for either sphere.} \quad (10.21)$$

The integral (10.18) has been evaluated numerically for the case of two equal-sized spheres ($K = 1$), the results obtained being presented in figure 6. Again these results agree with the asymptotic forms given by (10.19), (10.20) and (10.21) and it is observed that particle interaction has a large effect (but not as large as for particles lying on the r_3' axis) on the mean migration velocity, which is 51·2 % higher (for small d^*) than for an isolated particle for $\lambda_h = 1{\cdot}5$ $(d/(d+h) = 0{\cdot}4)$.

Motion of two spheres with their line of centres along the r_1' axis

The migration velocity experienced by sphere a may be obtained, for this situation, by letting $c^* = h^* = 0$ in (10.6), which then reduces to

$$v_l' = \frac{3}{4\pi} VRe \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{t+q}{t-q} (e^{-qd^*} - e^{-td^*})^2 (1 + K e^{-tk_1 s^*})\, dk_1\, dk_2. \quad (10.22)$$

Letting $\lambda_s = s^*/d^*$, the various asymptotic forms of this result may be obtained as:

(i) Both spheres close to wall $(s^*, d^* \to 0)$:

$$v_l' = VRe\,[\tfrac{3}{32} + \tfrac{3}{2}K(4+\lambda_s^2)^{-\frac{5}{2}}\{6\lambda_s(\nu/dV) + (2-\lambda_s^2)\}], \quad (10.23a)$$

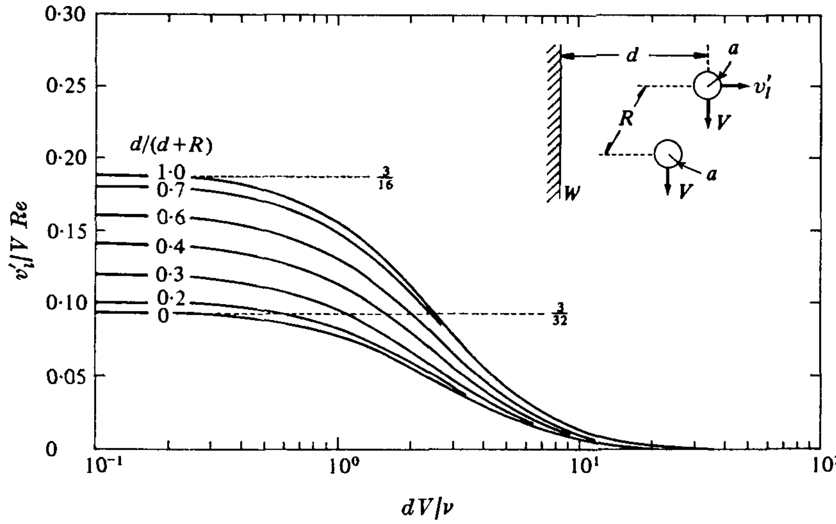

FIGURE 6. Lift velocity experienced by two spheres sedimenting with their line of centres along the r'_2 axis in the presence of a plane wall.

this giving the lift velocity on the lower or upper sphere according to whether λ_s is positive or negative. The mean lift velocity v'_{lm} is thus

$$v'_{lm} = VRe\{\tfrac{3}{32} + 3K(2-\lambda_s^2)/2(4+\lambda_s^2)^{\frac{5}{2}}\}. \tag{10.23b}$$

(ii) Spheres well separated from wall and from each other (s^*, $d^* \to \infty$):

$$v'_l = (a\nu/d^2)\{\tfrac{3}{8} + 3K/(4+\lambda_s^2)^{\frac{5}{2}}\} \quad \text{for either sphere.} \tag{10.24}$$

(iii) Spheres close to each other but at large distance from wall ($s^* \to 0$, $d^* \to \infty$):

$$v'_l = \tfrac{3}{8}(a\nu/d^2)(1+K) \quad \text{for either sphere.} \tag{10.25}$$

The integral (10.22) was evaluated numerically for the case of two equal-sized particles ($K = 1$) and the results are presented in figures 7(a) and 7(b). It is seen that there is agreement between these results and the asymptotic forms given by (10.23), (10.24) and (10.25). As $d^* \to 0$, the non-dimensional lift velocity v'_l/VRe is observed to tend to infinity except for the cases $\lambda_s = 0$ ($d/(d+s) = 1$) and $\lambda_s = \infty$ ($d/(d+s) = 0$). This effect, which is caused by the second term in (10.23a), gives rise to a lift velocity which for a fixed λ_s is proportional to aV/d. Thus, since this is proportional to the sedimentation velocity V and independent of ν, it is an effect which would occur even at zero Reynolds number and in fact one would expect such a lift velocity for this present situation. However if the mean lift velocity v'_{lm} of the spheres is calculated (see figure 7c) then v'_{lm}/VRe remains bounded as $d^* \to 0$ as is indicated in (10.23b). It is interesting that unlike the situations where the line of centres lies along the r'_2 or r'_3 axis, the mean lift velocity for the present case for $\lambda_s > \sqrt{2}$ ($0 < d/(d+s) < 0{\cdot}4142$) is less than for an isolated sphere (for small d^*), being a minimum at $\lambda_s = \sqrt{6}$ ($d/(d+s) = 0{\cdot}2899$) when the lift velocity reaches a value which is 20·2 % less than that for an isolated sphere. However for $\lambda_s > \sqrt{2}$ ($d/(d+s) > 0{\cdot}4142$) the mean lift velocity is

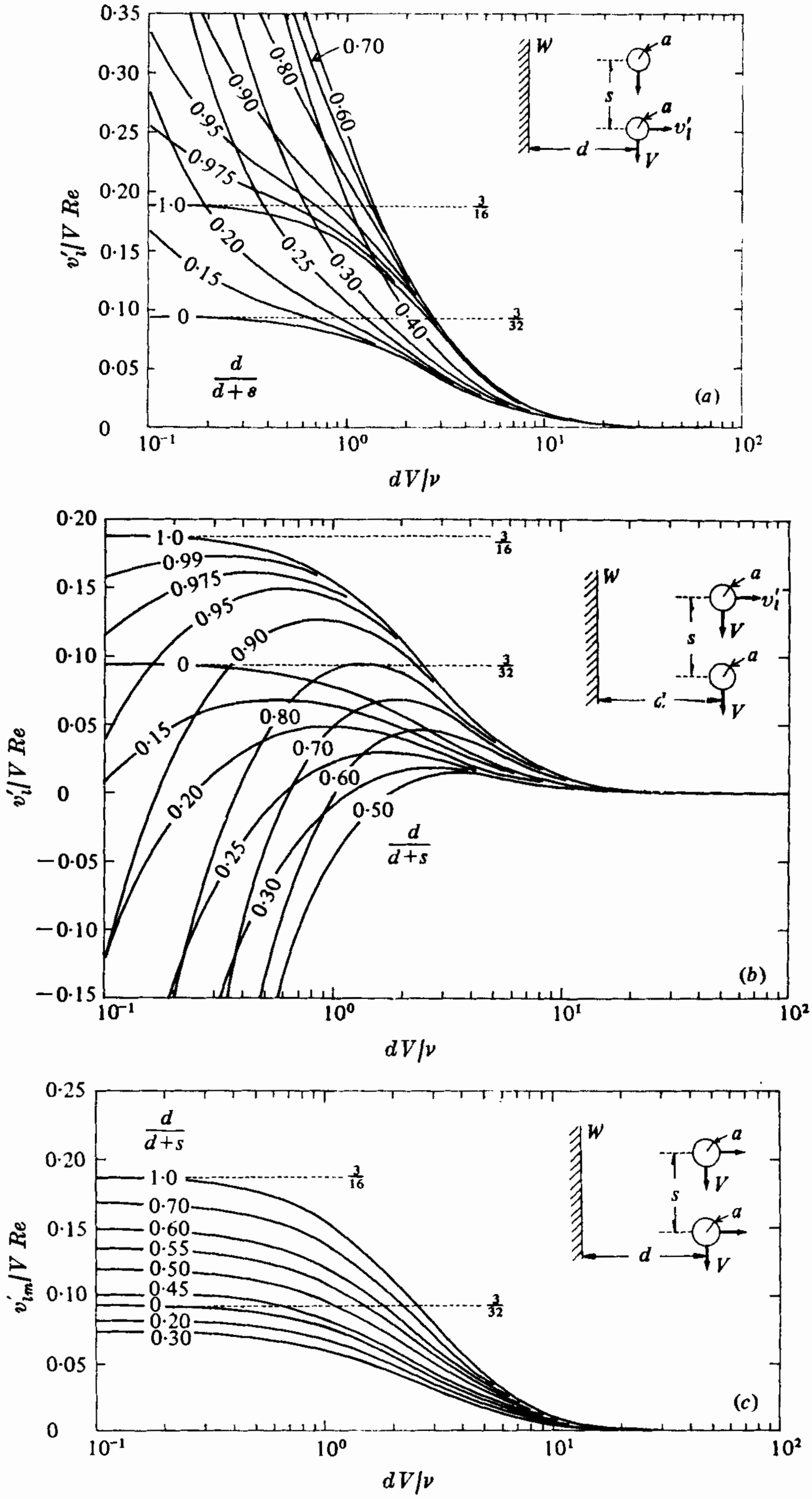

FIGURE 7. Lift velocity experienced by (*a*) the lower sphere and (*b*) the upper sphere of a pair of spheres sedimenting along their line of centres in a fluid bounded by a plane wall. (*c*) Their mean lift velocity.

higher than for an isolated sphere, being 28·6 % higher than that for an isolated sphere at $\lambda_s = 1\,(d/(d+s) = 0{\cdot}5)$. It is also noted that, whether the line of centres is along the r_1', r_2' or r_3' axis when they are close to each other and far from the wall, their lift velocity is double that of an isolated sphere (see (10.17c), (10.21) and (10.25)).

11. Sphere sedimenting in a stagnant fluid bounded by two plane walls

In this section, the motion of a single sphere of radius a sedimenting with a velocity V through a viscous quiescent fluid bounded by two infinite vertical plane walls is considered. A co-ordinate system (r_1', r_2', r_3') is chosen as in §2 (figure 1). The left-hand plane wall is at $r_3' = -d$ as before and the distance between the two parallel plane walls is l, the distances d and $l-d$ being assumed to be very much larger than the particle radius a. As before the flow is steady with the fluid velocity at infinity being $\mathbf{V} = (V, 0, 0)$. It is assumed that the conditions $a/d \ll 1$, $aV/\nu \ll 1$, and $lV/\nu = O(1)$ are satisfied. Thus, for the present problem $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$ satisfy (6.2) except that W now represents both walls.

In order to solve Oseen's equations with the point force we introduce, for convenience, a new co-ordinate system $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ whose origin is shifted from the sphere centre to the left-hand side wall so that

$$\bar{r}_1 = \tilde{r}_1, \quad \bar{r}_2 = \tilde{r}_2, \quad \bar{r}_3 = \tilde{r}_3 + d^*,$$

where $d^* = dV/\nu$ is the dimensionless value of the distance between the centre of the sphere and the left-hand side wall expressed in terms of the outer variables. The Fourier transforms $\mathbf{\Gamma}$ and Π satisfy (6.5) and (6.6) with the obvious replacement of $\delta(\tilde{r}_3)$ by $\delta(\bar{r}_3 - d^*)$, while the corresponding transformed boundary conditions are

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{on} \quad \bar{r}_3 = 0, \tag{11.1a}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{on} \quad \bar{r}_3 = l^*, \tag{11.1b}$$

where $l^* = lV/\nu$ is the dimensionless distance (l/a) between the two plane walls expressed in terms of the stretched outer variables.

These equations may be solved in a manner similar to that for the case considered in §6 to obtain

$$\Gamma_1 = -Aik_1 q^{-1} e^{-q\bar{r}_3} + G e^{-t\bar{r}_3} + Cik_1 q^{-1} e^{q\bar{r}_3} + H e^{t\bar{r}_3} - (3/4\pi)\{ik_1 q^{-1} e^{-q|\bar{r}_3 - d^*|} - (ik_1 - 1) t^{-1} e^{-t|\bar{r}_3 - d^*|}\}, \tag{11.2}$$

$$\Gamma_2 = -Aik_2 q^{-1} e^{-q\bar{r}_3} + E e^{-t\bar{r}_3} + Cik_2 q^{-1} e^{q\bar{r}_3} + F e^{t\bar{r}_3} - \frac{3}{4\pi}\{ik_2 q^{-1} e^{-q|\bar{r}_3 - d^*|} - ik_2 t^{-1} e^{-t|\bar{r}_3 - d^*|}\}, \tag{11.3}$$

$$\Gamma_3 = A e^{-q\bar{r}_3} + B e^{-t\bar{r}_3} + C e^{q\bar{r}_3} + D e^{t\bar{r}_3} + \frac{3}{4\pi} \operatorname{sgn}(\bar{r}_3 - d^*)\,(e^{-q|\bar{r}_3 - d^*|} - e^{-t|\bar{r}_3 - d^*|}), \tag{11.4}$$

$$\Pi = ik_1 q^{-1}(A e^{-q\bar{r}_3} - C e^{q\bar{r}_3} + \frac{3}{4\pi} e^{-q|\bar{r}_3 - d^*|}), \tag{11.5}$$

where A, B, C, D, E, F, G and H are constants of integration.

Using the boundary condition (11.1*a*) with equations (11.2), (11.3) and (11.4) one obtains the following relations:

$$A+B+C+D = (3/4\pi)\,(e^{-qd^*}-e^{-td^*}), \tag{11.6}$$

$$-Aik_2\,q^{-1}+E+Cik_2\,q^{-1}+F = (3/4\pi)\,(ik_2\,q^{-1}\,e^{-qd^*}-ik_2\,t^{-1}e^{-td^*}), \tag{11.7}$$

$$-Aik_1\,q^{-1}+G+Cik_1\,q^{-1}+H = (3/4\pi)\,(ik_1\,q^{-1}\,e^{-qd^*}-(ik_1-1)\,t^{-1}e^{-td^*}). \tag{11.8}$$

Similarly, from boundary condition (11.1*b*), one obtains

$$A\,e^{-ql^*}+B\,e^{-tl^*}+C\,e^{ql^*}+D\,e^{tl^*} = -\frac{3}{4\pi}\{e^{-q(l^*-d^*)}-e^{-t(l^*-d^*)}\}, \tag{11.9}$$

$$-Aik_2\,q^{-1}e^{-ql^*}+E\,e^{-tl^*}+Cik_2\,q^{-1}e^{ql^*}+F\,e^{tl^*} = \frac{3}{4\pi}\{ik_2\,q^{-1}\,e^{-q(l^*-d^*)} - ik_2\,t^{-1}\,e^{-t(l^*-d^*)}\}, \tag{11.10}$$

$$-Aik_1\,q^{-1}e^{-ql^*}+G\,e^{-tl^*}+Cik_1\,q^{-1}e^{ql^*}+H\,e^{tl^*} = \frac{3}{4\pi}\{ik_1\,q^{-1}\,e^{-q(l^*-d^*)} - (ik_1-1)\,t^{-1}e^{-t(l^*-d^*)}\}. \tag{11.11}$$

Furthermore, by substituting the values of Γ_1, Γ_2 and Γ_3 from relations (11.2)–(11.4) into the continuity equation (6.6), it is seen that

$$ik_1\,G+ik_2\,E = Bt, \quad ik_1\,H+ik_2\,F = -Dt. \tag{11.12}$$

Equations (11.6)–(11.12) constitute a system of algebraic linear equations in 8 unknowns A to H which may be solved, the solution so obtained being then substituted into (11.4) to give the value of Γ_3 as

$$\begin{aligned}\tfrac{4}{3}\pi s\Gamma_3 = {} & \{e^{qn}-e^{-qn}\}\{x^2(1-e^{yl^*})-y^2(1-e^{-xl^*})\}\\ & -\{e^{tn}-e^{-tn}\}\{x^2(1-e^{-yl^*})-y^2(1-e^{-xl^*})\}\\ & +2y(1-e^{-xl^*})\{qe^{t\bar{r}_3+qd^*}+t\,e^{q\bar{r}_3+td^*}\}\\ & +2x(1-e^{ylx^*})\{q\,e^{-t\bar{r}_3+qd^*}-t\,e^{q\bar{r}_3-td^*}\}\\ & +2y(1-e^{xl^*})\{q\,e^{-t\bar{r}_3-qd^*}+t\,e^{-q\bar{r}_3-td^*}\}\\ & +2x(1-e^{-yl^*})\{q\,e^{t\bar{r}_3-qd^*}-t\,e^{-q\bar{r}_3+td^*}\}\\ & +ik_1\{e^{qm}(e^{-xl^*}-e^{yl^*})+e^{-qm}(e^{xl^*}-e^{-yl^*})\}\\ & +ik_1\{e^{tm}(e^{-xl^*}-e^{-yl^*})+e^{-tm}(e^{xl^*}-e^{yl^*})\}\\ & +s(|n|/n)\{e^{-q|n|}-e^{-t|n|}\},\end{aligned} \tag{11.13}$$

where $s = x^2(e^{-ql^*}-e^{-tl^*})\,(e^{ql^*}-e^{tl^*}) - y^2(e^{-ql^*}-e^{tl^*})\,(e^{ql^*}-e^{-tl^*})$,

$n = (\bar{r}_3-d^*), \quad m = (\bar{r}_3+d^*), \quad x = t+q, \quad y = t-q.$

Substituting the results (11.13) and (7.11) into (8.4) and (8.5) and evaluating the resulting integrand for $\bar{r}_3 \to d^*$ yields

$$\begin{aligned}\mathbf{v}'_l = \frac{3}{2\pi}\,VRe\,\mathbf{e}_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ik_1}{\lambda}\{&\cosh[(t+q)\,d^*]-\cosh[(t-q)\,d^*] + \cosh[(t-q)\,(l^*-d^*)]\\ &-\cosh[(t+q)\,(l^*-d^*)]+\sinh ql^*\sinh[t(l^*-2d^*)]+\sinh tl^*\\ &\times\sinh[q(l^*-2d^*)]\}\,dk_1\,dk_2,\end{aligned} \tag{11.14}$$

where $\lambda = 4tq-(t+q)^2\cosh[(t-q)\,l^*]+(t-q)^2\cosh[(t+q)\,l^*]$.

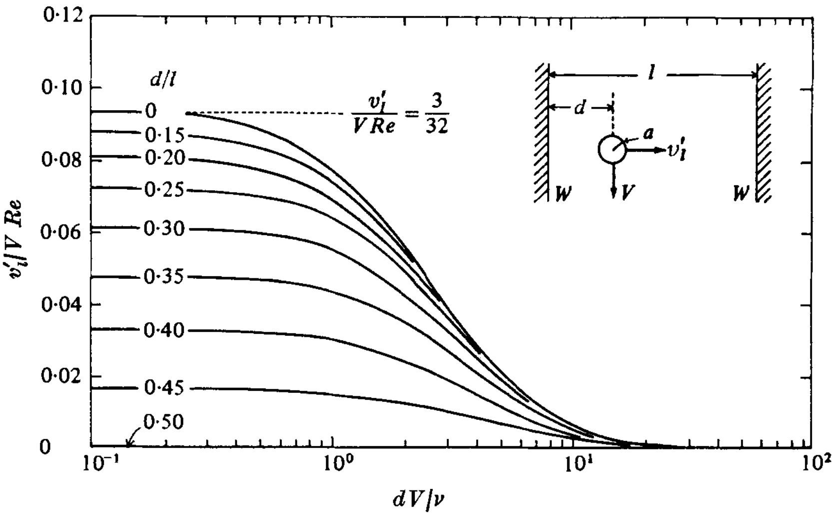

FIGURE 8. Lift velocity v_l' experienced by a spherical particle sedimenting in a fluid bounded by two vertical plane walls.

For the special case of a sphere lying midway between the two plane walls (i.e. $l = 2d$), the integrand in (11.14) reduces to zero and the sphere experiences no lift force as expected. In general, however, the integral appearing in (11.14) cannot be determined analytically and a numerical procedure must be used. Such a numerical evaluation of (11.14) has been performed on an IBM 360 computer, the results obtained being presented in figure 8, where the migration velocity normalized with respect to VRe is plotted as a function of the variable $d^* = dV/\nu$ for different values of the parameter $(d^*/l^*) = (d/l)$, i.e. for different positions of the spherical particle relative to the walls. Only the values of (d/l) between 0 and 0·5 are shown in this graph since the motion of the particle is symmetrical about $(d/l) = 0{\cdot}5$. The positive values of v_l'/VRe imply that the particle moves away from the walls until it reaches an equilibrium position midway between the two plane walls. It is noted that for small values of d^* (i.e. $d^* < 0{\cdot}2$) the lift velocity tends to a constant for each value of (d/l).

The migration velocity experienced by a spherical particle sedimenting in a fluid bounded by a single plane wall (i.e. $(d/l) = 0$) has been studied by Cox & Hsu (1976) on the basis of a completely different theory. It was assumed, in their analysis, that the particle was located close enough to the wall to be inside the inner region of expansion. Their result, namely $v_l'/VRe = \frac{3}{32}$, is shown in this graph and is seen to be valid for values of $d^* < 0{\cdot}2$ only.

It should also be noted that the value of the migration velocity as calculated here is independent of whether the sphere rotates or not as long as its angular velocity is $o(Re)$ as $Re \to 0$. As has been shown, this is the case for a freely rotating sphere [and also obviously for a sphere prevented from rotating by an external torque].

12. Experimental

The radial migration phenomenon was studied by observing the trajectory of single rigid spherical particles released into a stagnant, Newtonian fluid bounded by parallel vertical plane walls.

The test section used for this purpose consisted of a transparent, 9 feet long, vertical duct, with a rectangular internal cross-section of size 1·2 in. (between the narrowly spaced walls) by 7·5 in. (between the widely spaced walls) resulting in an aspect ratio of 6. With such an aspect ratio the effects of the two widely spaced walls on the particle were expected to be negligible.

The particle when placed in the mid-plane between the widely spaced walls was observed to remain in the mid-plane while it migrated towards or away from the narrowly spaced walls. This migration was observed by viewing along a direction parallel to the narrowly spaced walls. At various positions along the length of the channel, a set of hairlines lightly etched on the inside front surface of the channel aided in locating particles in the narrow direction. These hairlines were coloured with indelible ink for improved visibility.

The fluid in the test solution consisted of 69·7 % glycerol and 30·3 % distilled water by weight, resulting in a specific gravity of 1·181. Because the viscosity of the test fluid is sensitive to temperature changes as near isothermal conditions as can be achieved are required within the test section. In the present study this was done by circulating water in a cooling jacket surrounding the test section. This resulted in a nearly uniform temperature of the test fluid which was measured directly with thermometers located at the top and at the bottom of the test section. Further it was found that the variation of temperature during an experiment was negligible.

The five spherical resin particles used in the experiments were selected from a mixture of particles of different sizes. The diameter of the particles, ranging from $2{\cdot}47 \times 10^{-2}$ in. to $4{\cdot}4 \times 10^{-2}$ in., was determined from micrometer measurements across fifty different diameters for each particle. All particles were soaked for several weeks before use to avoid diameter changes accompanying their swelling resulting from water absorption.

A vacuum release system, in which a particle held onto the tip of a hypodermic needle by the suction provided by a filter pump and released by breaking the vacuum, was found to be suitable for the controlled release of a particle. This particle injector was placed above the test section and was designed to inject a particle at any required position. Particles were supplied to this needle through a sealable door.

The trajectory of the particle was recorded by a camera travelling along vertical I-beams (7 feet long) parallel to the duct thus permitting the shaft-driven camera mount to follow the particle over the entire length of the test section. Using two $\frac{1}{16}$ h.p., 250 r.p.m. electric motors connected to the driving shaft by a 2:1 gear reduction unit, camera speeds of 0 to 0·01 ft/s could be obtained. By matching the camera speed to that of a particle in the section, the latter can be maintained in the centre of the field of view and examined for as long as was desired. Limit switches were used to reverse the direction of the camera at each end of its travel.

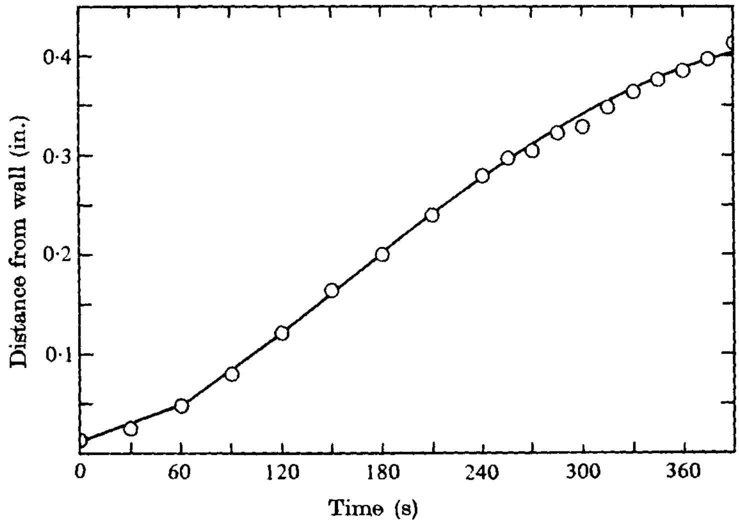

FIGURE 9. Experimentally observed position of sphere:
$\nu = 2{\cdot}44 \times 10^{-4}\ \mathrm{ft^2/s}$, $a = 0{\cdot}0363$ in.,
$V = 5{\cdot}81 \times 10^{-3}\ \mathrm{ft/s}$, $Re = 0{\cdot}072$.

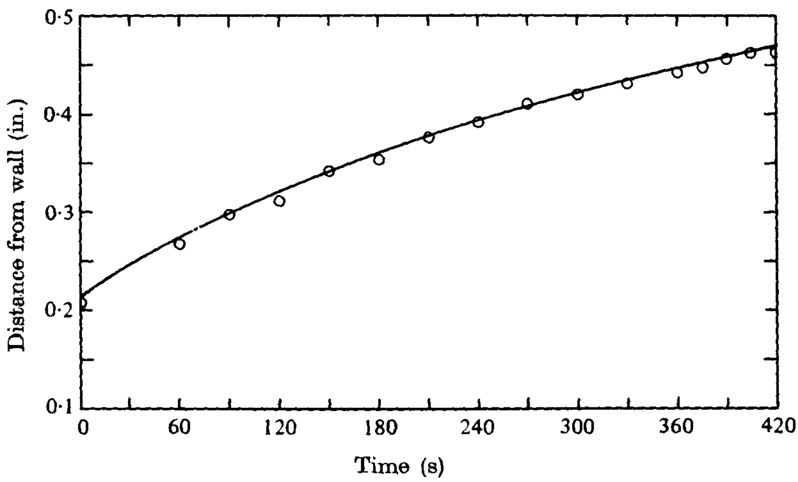

FIGURE 10. Experimentally observed position of sphere:
$\nu = 1{\cdot}75 \times 10^{-1}\ \mathrm{ft^2/s}$, $a = 0{\cdot}0311$ in.,
$V = 6{\cdot}28 \times 10^{-3}\ \mathrm{ft/s}$, $Re = 0{\cdot}093$.

The camera used in the experimental study was a 35 mm Nikon model F which was aligned normal to the front face of the channel and was rigidly attached to the metal rail described in the above paragraph. The camera lens, a micro-Nikhor auto 1:35 $F = 55$ mm, produced a subject area of 1·93 in. by 1·45 in. All data were recorded with Kodak high speed ektachrome film (ASA-160) at setting f/32.

The particle motion was analysed by projecting the slides onto a screen using a ciné Kodak Ektagraphic projector, a constant magnification factor of 14·5 being

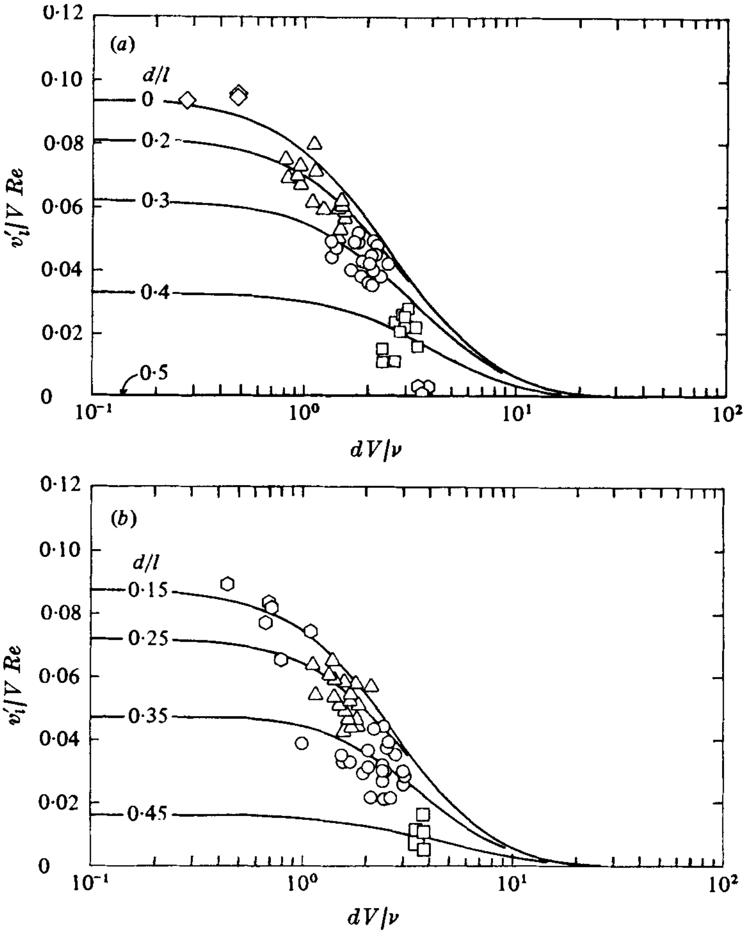

FIGURE 11. Migration velocity versus position for the sedimenting spherical particle. (*a*) ◇, d/l = 0·1; △, d/l = 0·2; ○, d/l = 0·3; □, d/l = 0·4; ⬡, d/l = 0·5. (*b*) ⬡, d/l = 0·15; △, d/l = 0·25; ○, d/l = 0·35; □, d/l = 0·45.

achieved. Particle trajectories and velocities were determined from successive slide measurements.

Complete details on the experimental apparatus, procedure, and method of data reduction are given by Vasseur (1973).

A few examples of the experimental data obtained in this study are shown graphically in figures 9 and 10. These graphs are typical plots of the radial position of a particle from the wall, in inches, as a function of time, in seconds, for various experimental conditions. Solid lines, in these plots, represent smooth curves drawn through the experimental points. These plots clearly demonstrate the migration phenomenon and it is seen that the migration, which is relatively large near the wall, decreases rapidly as the particle approaches the axis of the test section. It is also interesting to note that according to figure 9 a particle sedimenting in the neighbourhood of a plane wall migrates away from the wall with a constant velocity. Such a behaviour is qualitatively in agreement with the present theory, which predicts that a particle located in the neighbourhood of a plane wall migrates with a constant velocity $v_l' = \frac{3}{32} VRe$. Although the data are not sufficient to settle the matter definitely, it seems that there exists a discrepancy between the theoretical constant $\frac{3}{32}$ and the experimentally measured constant. However this is not surprising since the present theory is not expected to be valid when the particle is so close to the wall that it almost makes contact with it.

Migration velocities were calculated from the trajectory data obtained in the experimental investigation by numerically fitting a curve through consecutive points on the radial position–time diagram and evaluating the slope of this curve. In this manner the radial velocity of the particles was calculated as a function of time. Then from the plots of the radial position versus time, it is possible to obtain the migration velocity as a function of the radial position. Typical results are presented in figure 11, in which the calculated migration velocity v_l', normalized with respect to VRe, is presented as a function of the dimensionless distance between the wall and the particle centre $d^* = (dV/\nu)$ for various values of the ratio (d/l) of the distance between the particle centre and the wall and the distance between the two walls. The theoretically predicted curves appear on the plots as solid lines and it is seen that the theoretically predicted values of the migration velocity v_l' agree well with the measured values.

13. Conclusions

(*a*) For an isolated sphere sedimenting in a fluid bounded by a vertical plane wall, it was found that:

(i) the sphere always migrates away from the wall;

(ii) the drag force on the sphere is increased by the presence of the wall when the sphere is near the wall (d^* is small) but is decreased when the sphere is far from the wall (d^* is large).

(*b*) For an isolated sphere sedimenting in a fluid bounded by two vertical plane walls:

(i) the particle migrates away from the walls until it reaches an equilibrium position mid-way between the walls;

(ii) the experimental study of the migration of rigid spherical particles sedimenting in a stagnant viscous fluid, although not extensive, provides new information about the migration phenomena. The observed migration rates, obtained by measuring the trajectories of the particles, are found to be in good agreement with those predicted by the present theory. The discrepancy between the theory and the experiments, observed in the vicinity of the wall, is due to the fact that the present theory is not expected to be valid when the particle is too close to the wall when the required condition $a/d \ll 1$ is not satisfied.

(*c*) For a pair of equal spheres sedimenting in an unbounded fluid:

(i) there is a repulsion between them if they sediment with their line of centres horizontal, their sedimentation velocity being higher than that of an isolated sphere;

(ii) the trailing sphere catches up the leading sphere if their line of centres is vertical, their mean sedimentation velocity being higher than that of an isolated sphere;

(iii) if their line of centres is neither horizonta nor vertical, then there is a mean horizontal migration of the spheres.

(*d*) For a pair of equal spheres sedimenting in a fluid bounded by a vertical plane wall:

(i) the mean migration velocity of a pair of spheres is away from the wall even though an individual sphere of a pair may move towards the wall;

(ii) the mean migration velocity is considerably different from that of an isolated sphere even when the sphere separation is of the order of their distance from the wall;

(iii) when the two spheres are close to each other, their mean migration velocity is twice that of an isolated sphere. However at large separation with their line of centres in the vertical direction their mean migration velocity can be less than that of an isolated sphere by up to 20·2 %.

These results indicate that the effect of a second particle on the migration velocity experienced by a particle is considerable. Thus it is expected that, for the case of a cloud of sedimenting particles or a suspension of sedimenting particles in a fluid, each particle will experience a migration velocity which will be considerably different from that of an isolated particle.

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